VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2008

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Figures									Letter
Words									

MATHEMATICAL METHODS (CAS)

Written examination 2

Monday 10 November 2008

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 24 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

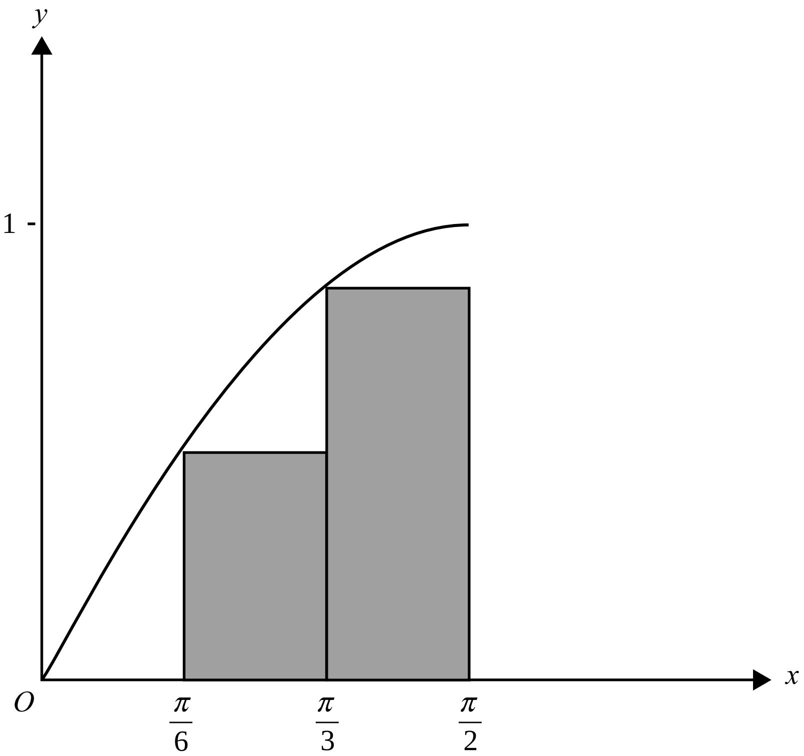
Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1



The area under the curve $y = \sin(x)$ between $x = 0$ and $x = \frac{\pi}{2}$ is approximated by two rectangles as shown. This approximation to the area is

A. 1

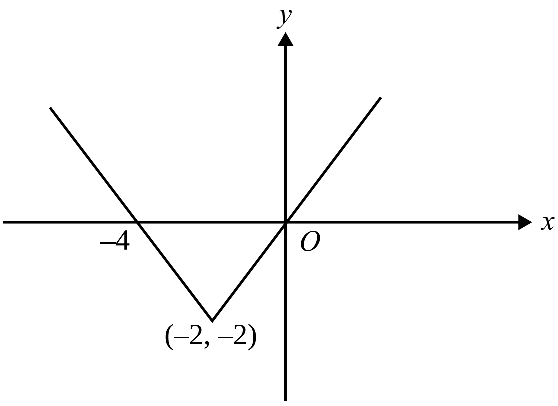
B. $\frac{\pi}{2}$

C. $\frac{(\sqrt{3}+1)\pi}{12}$

D. 0.5

E. $\frac{(\sqrt{3}+1)\pi}{6}$

Question 2



The rule of the function whose graph is shown is

A. $y = |x| - 4$

B. $y = |x - 2| + 2$

C. $y = |x + 2| - 2$

D. $y = |2 - x| - 2$

E. $y = |2 + x| + 2$

Question 3

The average value of the function with rule $f(x) = \log_e(3x + 1)$ over the interval [0, 2] is

A. $\dfrac{\log_e(7)}{2}$

B. $\log_e(7)$

C. $\dfrac{7\log_e(7)}{3} - 2$

D. $\dfrac{7\log_e(7) - 6}{6}$

E. $\dfrac{35\log_e(7) - 12}{18}$

Question 4

If $\int_1^3 f(x)dx = 5$, then $\int_1^3 (2f(x) - 3)dx$ is equal to

A. 4

B. 5

C. 7

D. 10

E. 16

Question 5

Let X be a discrete random variable with a binomial distribution. The mean of X is 1.2 and the variance of X is 0.72.

The values of n (the number of independent trials) and p (the probability of success in each trial) are

A. $n = 4, \quad p = 0.3$

B. $n = 3, \quad p = 0.6$

C. $n = 2, \quad p = 0.6$

D. $n = 2, \quad p = 0.4$

E. $n = 3, \quad p = 0.4$

Question 6

The simultaneous linear equations

$$ax + 3y = 0$$
$$2x + (a + 1)y = 0$$

where a is a real constant, have infinitely many solutions for

A. $a \in R$

B. $a \in \{-3, 2\}$

C. $a \in R \setminus \{-3, 2\}$

D. $a \in \{-2, 3\}$

E. $a \in R \setminus \{-2, 3\}$

Question 7

The inverse of the function $f: R^+ \to R,\ f(x) = \frac{1}{\sqrt{x}} - 3$ is

A. $f^{-1}: R^+ \to R \qquad f^{-1}(x) = (x + 3)^2$

B. $f^{-1}: R^+ \to R \qquad f^{-1}(x) = \frac{1}{x^2} + 3$

C. $f^{-1}: (3, \infty) \to R \qquad f^{-1}(x) = \frac{-1}{(x-3)^2}$

D. $f^{-1}: (-3, \infty) \to R \qquad f^{-1}(x) = \frac{1}{(x+3)^2}$

E. $f^{-1}: (-3, \infty) \to R \qquad f^{-1}(x) = -\frac{1}{x^2} - 3$

Question 8

The graph of the function $f: D \to R,\ f(x) = \frac{x-3}{2-x}$, where D is the maximal domain has asymptotes

A. $x = 3, \quad y = 2$

B. $x = -2, \quad y = 1$

C. $x = 1, \quad y = -1$

D. $x = 2, \quad y = -1$

E. $x = 2, \quad y = 1$

Question 9

The transformation $T: R^2 \to R^2$ with rule

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

maps the curve with equation $y = x^3$ to the curve with equation

A. $y = \dfrac{-(x-1)^3}{32} + 3$

B. $y = \dfrac{-(4x+1)^3 + 3}{2}$

C. $y = \dfrac{-(x+1)^3}{32} - 3$

D. $y = \dfrac{(1-x)^3}{64} - 3$

E. $y = \dfrac{(4x-1)^3 + 3}{2}$

Question 10

The range of the function $f: \left[\dfrac{\pi}{8}, \dfrac{\pi}{3}\right) \to R,\ f(x) = 2\sin(2x)$ is

A. $\left(\sqrt{2}, \sqrt{3}\right]$

B. $\left[\sqrt{2}, 2\right)$

C. $\left[\sqrt{2}, 2\right]$

D. $\left(\sqrt{2}, \sqrt{3}\right)$

E. $\left[\sqrt{2}, \sqrt{3}\right)$

Question 11

The probability density function for the continuous random variable X is given by

$$f(x) = \begin{cases} |1-x| & \text{if } 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

The probability that $X < 1.5$ is equal to

A. 0.125
B. 0.375
C. 0.5
D. 0.625
E. 0.75

Question 12

Let $f: R \to R,\ f(x) = e^x + e^{-x}$.

For all $u \in R,\ f(2u)$ is equal to

A. $f(u) + f(-u)$

B. $2f(u)$

C. $(f(u))^2 - 2$

D. $(f(u))^2$

E. $(f(u))^2 + 2$

Question 13

According to a survey, 30% of employed women have never been married.

If 10 employed women are selected at random, the probability (correct to four decimal places) that at least 7 have never been married is

A. 0.0016

B. 0.0090

C. 0.0106

D. 0.9894

E. 0.9984

Question 14

The minimum number of times that a fair coin can be tossed so that the probability of obtaining a head on each trial is less than 0.0005 is

A. 8

B. 9

C. 10

D. 11

E. 12

Question 15

The sample space when a fair die is rolled is $\{1, 2, 3, 4, 5, 6\}$, with each outcome being equally likely.

For which of the following pairs of events are the events independent?

A. $\{1, 2, 3\}$ and $\{1, 2\}$

B. $\{1, 2\}$ and $\{3, 4\}$

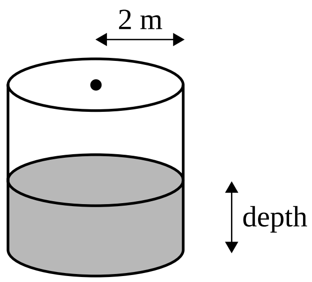
C. $\{1, 3, 5\}$ and $\{1, 4, 6\}$

D. $\{1, 2\}$ and $\{1, 3, 4, 6\}$

E. $\{1, 2\}$ and $\{2, 4, 6\}$

Question 16

Water is being poured into a long cylindrical storage tank of radius 2 metres, with its circular base on the ground, at a rate of 2 cubic metres per second.



The rate of change of the depth of the water, in metres per second, in the tank is

A. $\frac{1}{8\pi}$

B. $\frac{1}{4\pi}$

C. $\frac{1}{2\pi}$

D. 2π

E. 8π

Question 17

The graph of the function $f(x) = e^{2x} - 2$ intersects the graph of $g(x) = e^x$ where

A. $x = -1$

B. $x = \log_e(2)$

C. $x = 2$

D. $x = \frac{1+\sqrt{7}}{2}$

E. $x = \log_e\left(\frac{1+\sqrt{7}}{2}\right)$

Question 18

Let $f: \left[0, \frac{\pi}{2}\right] \to R,\ f(x) = \sin(4x) + 1$. The graph of f is transformed by a reflection in the x-axis followed by a dilation of factor 4 from the y-axis.

The resulting graph is defined by

A. $g: \left[0, \frac{\pi}{2}\right] \to R \qquad g(x) = -1 - 4\sin(4x)$

B. $g: [0, 2\pi] \to R \qquad g(x) = -1 - \sin(16x)$

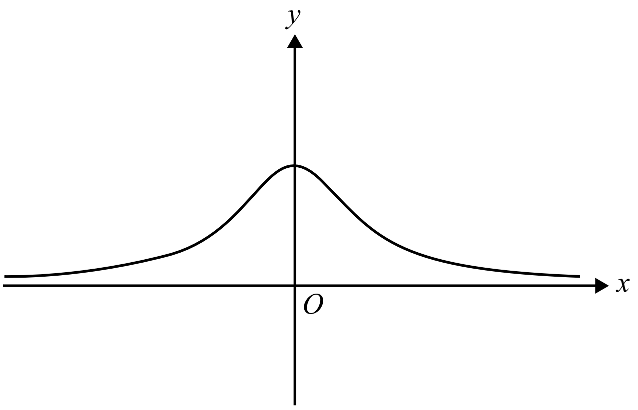
C. $g: \left[0, \frac{\pi}{2}\right] \to R \qquad g(x) = 1 - \sin(x)$

D. $g: [0, 2\pi] \to R \qquad g(x) = 1 - \sin(4x)$

E. $g: [0, 2\pi] \to R \qquad g(x) = -1 - \sin(x)$

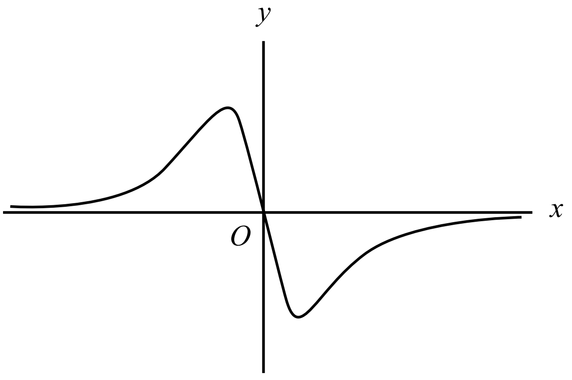
Question 19

The graph of a function f is shown below.

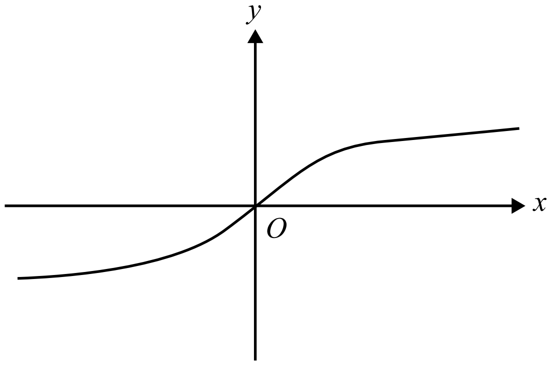


The graph of an **antiderivative** of f could be

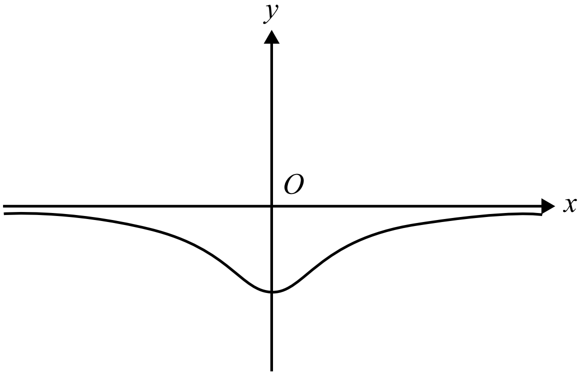
A.



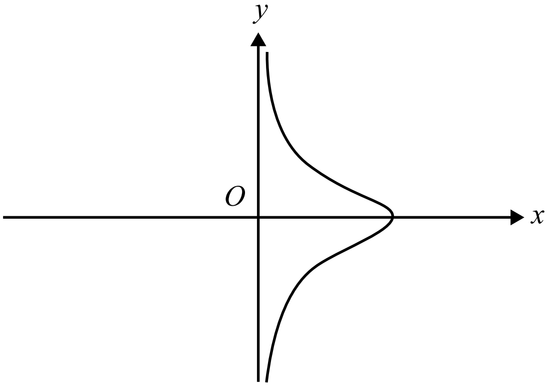
B.



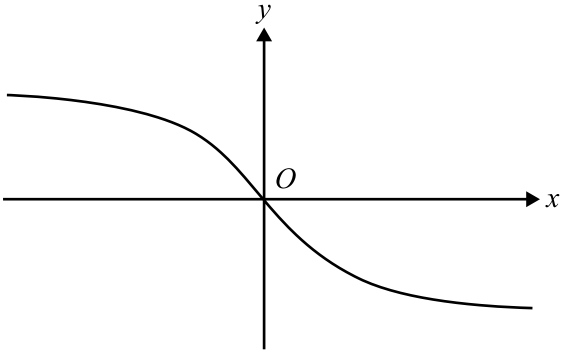
C.



D.



E.



Question 20

The function $f: B \to R$ with rule $f(x) = 4x^3 + 3x^2 + 1$ will have an inverse function for

A. $B = R$

B. $B = \left(\frac{1}{2}, \infty\right)$

C. $B = \left(-\infty, \frac{1}{2}\right]$

D. $B = \left(-\infty, \frac{1}{2}\right)$

E. $B = \left[-\frac{1}{2}, \infty\right)$

Question 21

The graph of $y = x^3 - 12x$ has turning points where $x = 2$ and $x = -2$.
The graph of $y = |\, x^3 - 12x \,|$ has a positive gradient for

A. $x \in R$

B. $x \in \{x : x < -2\} \cup \{x : x > 2\}$

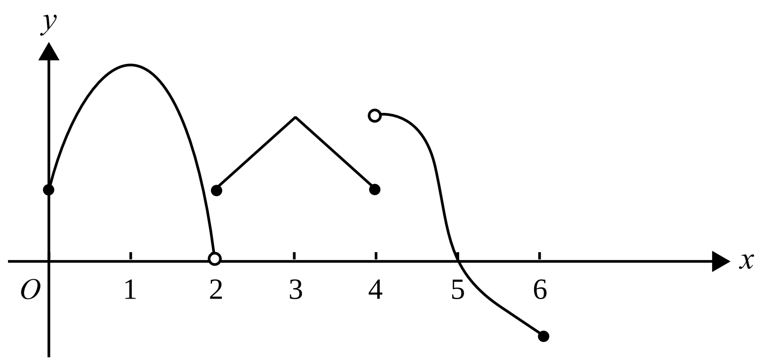
C. $x \in \left\{x : x < -2\sqrt{3}\right\} \cup \left\{x : x > 2\sqrt{3}\right\}$

D. $x \in \left\{x : -2\sqrt{3} < x < -2\right\} \cup \{x : 0 < x < 2\} \cup \left\{x : x > 2\sqrt{3}\right\}$

E. $x \in \{x : -2 < x < 0\} \cup \left\{x : 2 < x < 2\sqrt{3}\right\} \cup \left\{x : x > 2\sqrt{3}\right\}$

Question 22

The graph of the function f with domain [0, 6] is shown below.



Which one of the following is **not** true?

A. The function is not continuous at $x = 2$ and $x = 4$.

B. The function exists for all values of x between 0 and 6.

C. $f(x) = 0$ for $x = 2$ and $x = 5$.

D. The function is positive for $x \in [0, 5)$.

E. The gradient of the function is not defined at $x = 4$.

END OF SECTION 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Sharelle is the goal shooter for her netball team. During her matches, she has many attempts at scoring a goal.

Assume that each attempt at scoring a goal is independent of any other attempt. In the long term, her scoring rate has been shown to be 80% (that is, 8 out of 10 attempts to score a goal are successful).

a. **i.** What is the probability, correct to four decimal places, that her first 8 attempts at scoring a goal in a match are successful?

ii. What is the probability, correct to four decimal places, that exactly 6 of her first 8 attempts at scoring a goal in a match are successful?

1 + 2 = 3 marks

Assume instead that the success of an attempt to score a goal depends only on the success or otherwise of her previous attempt at scoring a goal.

If an attempt at scoring a goal in a match is successful, then the probability that her next attempt at scoring a goal in the match is successful is 0.84. However, if an attempt at scoring a goal in a match is unsuccessful, then the probability that her next attempt at scoring a goal in the match is successful is 0.64.

Her first attempt at scoring a goal in a match is successful.

b. i. What is the probability, correct to four decimal places, that her next 7 attempts at scoring a goal in the match will be successful?

ii. What is the probability, correct to four decimal places, that exactly 2 of her next 3 attempts at scoring a goal in the match will be successful?

iii. What is the probability, correct to four decimal places, that her 8th attempt at scoring a goal in the match will be successful?

iv. In the long term, what percentage of her attempts at scoring a goal are successful?

1 + 3 + 3 + 1 = 8 marks

The time in hours that Sharelle spends training each day is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \dfrac{1}{64}(6-x)(x-2)(x+2) & \text{if} \quad 2 \le x \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

c. **i.** Sketch the probability density function, and label the local maximum with its coordinates, correct to two decimal places.

ii. What is the probability, correct to four decimal places, that Sharelle spends less than 3 hours training on a particular day?

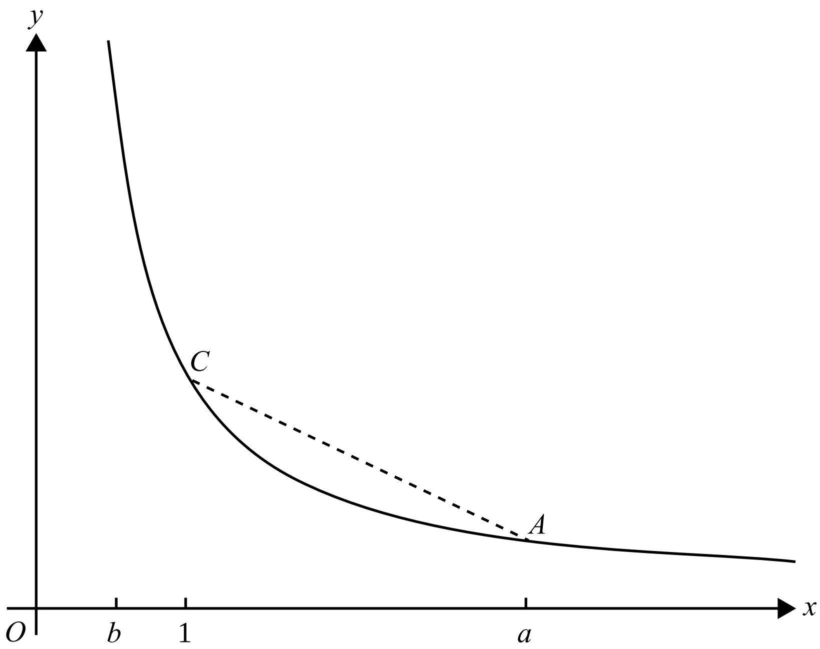
iii. What is the mean time (in hours), correct to four decimal places, that she spends training each day?

2 + 2 + 2 = 6 marks

Total 17 marks

Question 2

The diagram below shows part of the graph of the function $f: R^+ \to R, \quad f(x) = \frac{7}{x}$.



The line segment CA is drawn from the point $C\big(1,\ f(1)\big)$ to the point $A\big(a,\ f(a)\big)$ where $a > 1$.

a. **i.** Calculate the gradient of CA in terms of a.

ii. At what value of x between 1 and a does the tangent to the graph of f have the same gradient as CA?

1 + 2 = 3 marks

b. **i.** Calculate $\int_1^e f(x)dx$.

ii. Let b be a positive real number less than one. Find the exact value of b such that $\int_b^1 f(x)dx$ is equal to 7.

1 + 2 = 3 marks

c. **i.** Express the area of the region bounded by the line segment CA, the x-axis, the line $x = 1$ and the line $x = a$ in terms of a.

ii. For what exact value of a does this area equal 7?

iii. Using the value for a determined in **c.ii.**, explain in words, without evaluating the integral, why $\int_1^a f(x)dx < 7$.

Use this result to explain why $a < e$.

2 + 2 + 1 = 5 marks

d. Find the exact values of m and n such that $\int_1^{mn} f(x)dx = 3$ and $\int_1^{\frac{m}{n}} f(x)dx = 2$.

2 marks

Total 13 marks

Working space

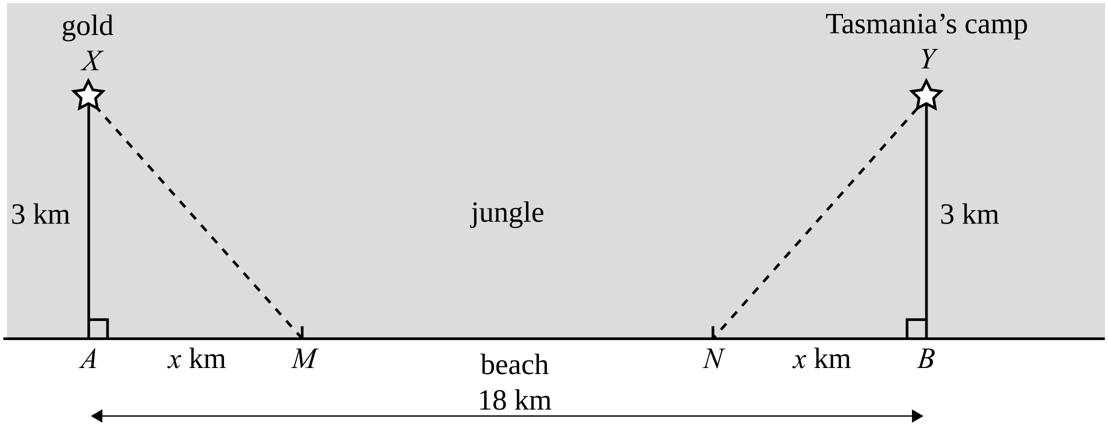
Question 3

Tasmania Jones is in the jungle, digging for gold. He finds the gold at X which is 3 km from a point A.

Point A is on a straight beach.

Tasmania's camp is at Y which is 3 km from a point B. Point B is also on the straight beach.

$AB = 18$ km and $AM = NB = x$ km and $AX = BY = 3$ km.



While he is digging up the gold, Tasmania is bitten by a snake which injects toxin into his blood. After he is bitten, the concentration of the toxin in his bloodstream increases over time according to the equation

$$y = 50 \log_e(1 + 2t)$$

where y is the concentration, and t is the time in hours after the snake bites him.

The toxin will kill him if its concentration reaches 100.

a. Find the time, to the nearest minute, that Tasmania has to find an antidote (that is, a cure for the toxin).

2 marks

Tasmania has an antidote to the toxin at his camp. He can run through the jungle at 5 km/h and he can run along the beach at 13 km/h.

b. Show that he will not get the antidote in time if he runs directly to his camp through the jungle.

1 mark

In order to get the antidote, Tasmania runs through the jungle to M on the beach, runs along the beach to N and then runs through the jungle to the camp at Y. M is x km from A and N is x km from B. (See diagram.)

c. Show that the time taken to reach the camp, T hours, is given by

$$T = 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$$

2 marks

d. Find the value of x which allows Tasmania to get to his camp in the minimum time.

2 marks

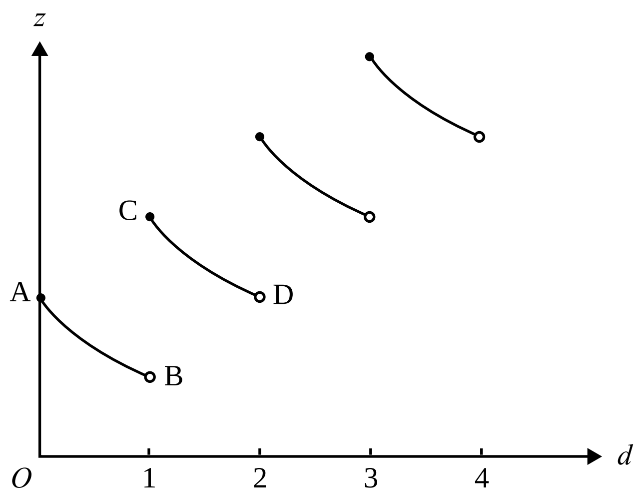
e. Show that he gets to his camp in time to get the antidote.

1 mark

At his camp, Tasmania Jones takes a capsule containing 16 units of antidote to the toxin. After taking the capsule the quantity of antidote in his body decreases over time.

At exactly the same time on successive days, he takes another capsule containing 16 units of antidote and again the quantity of antidote decreases in his body.

The graph of the quantity of antidote z units in his body at time d days after taking the first capsule looks like this. Each section of the curve has exactly the same shape as curve AB.



The equation of the curve AB is $z = \dfrac{16}{d+1}$

f. Write down the coordinates of the points A and C.

2 marks

g. Find the equation of the curve CD.

2 marks

Tasmania will no longer be affected by the snake toxin when he first has 50 units of the antidote in his body.

h. Assuming he takes a capsule at the same time each day, on how many days does he need to take a capsule so that he will no longer be affected by the snake toxin?

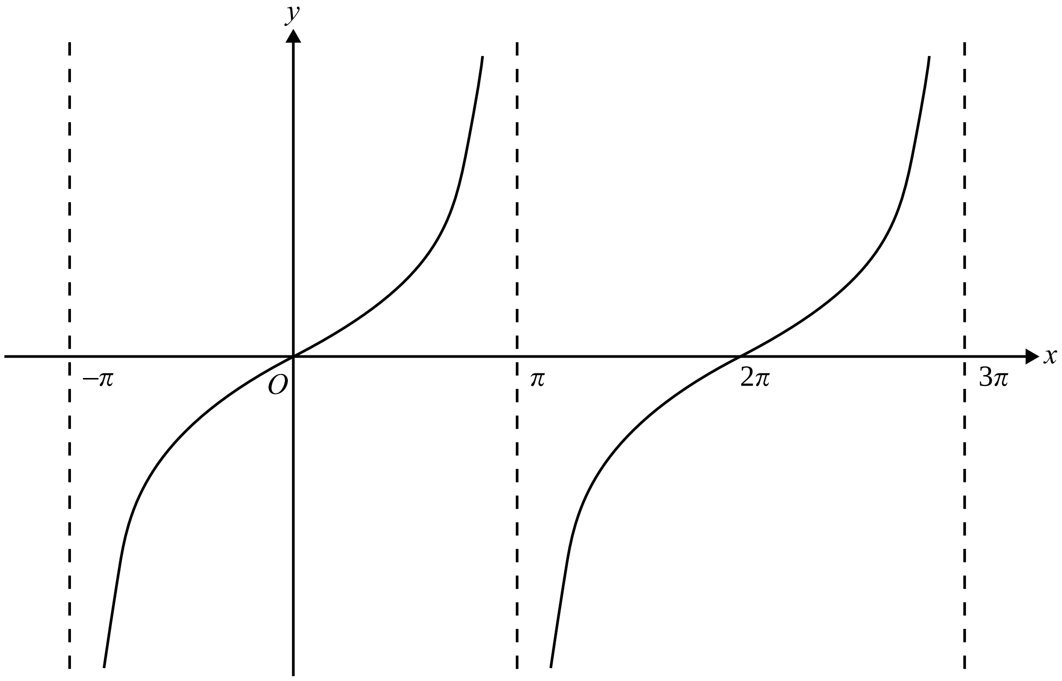
1 mark

Total 13 marks

Working space

Question 4

The graph of $f: (-\pi, \pi) \cup (\pi, 3\pi) \to R,\ f(x) = \tan\left(\frac{x}{2}\right)$ is shown below.



a. **i.** Find $f'\left(\frac{\pi}{2}\right)$.

ii. Find the equation of the **normal** to the graph of $y = f(x)$ at the point where $x = \frac{\pi}{2}$.

iii. Sketch the graph of this normal on the axes above. Give the exact axis intercepts.

1 + 2 + 3 = 6 marks

b. Find the exact values of $x \in (-\pi, \pi) \cup (\pi, 3\pi)$ such that $f'(x) = f'\left(\frac{\pi}{2}\right)$.

2 marks

Let $g(x) = f(x - a)$.

c. Find the exact value of $a \in (-1, 1)$ such that $g(1) = 1$.

2 marks

Let $h : (-\pi, \pi) \cup (\pi, 3\pi) \to R$, $h(x) = \sin\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right) + 2$.

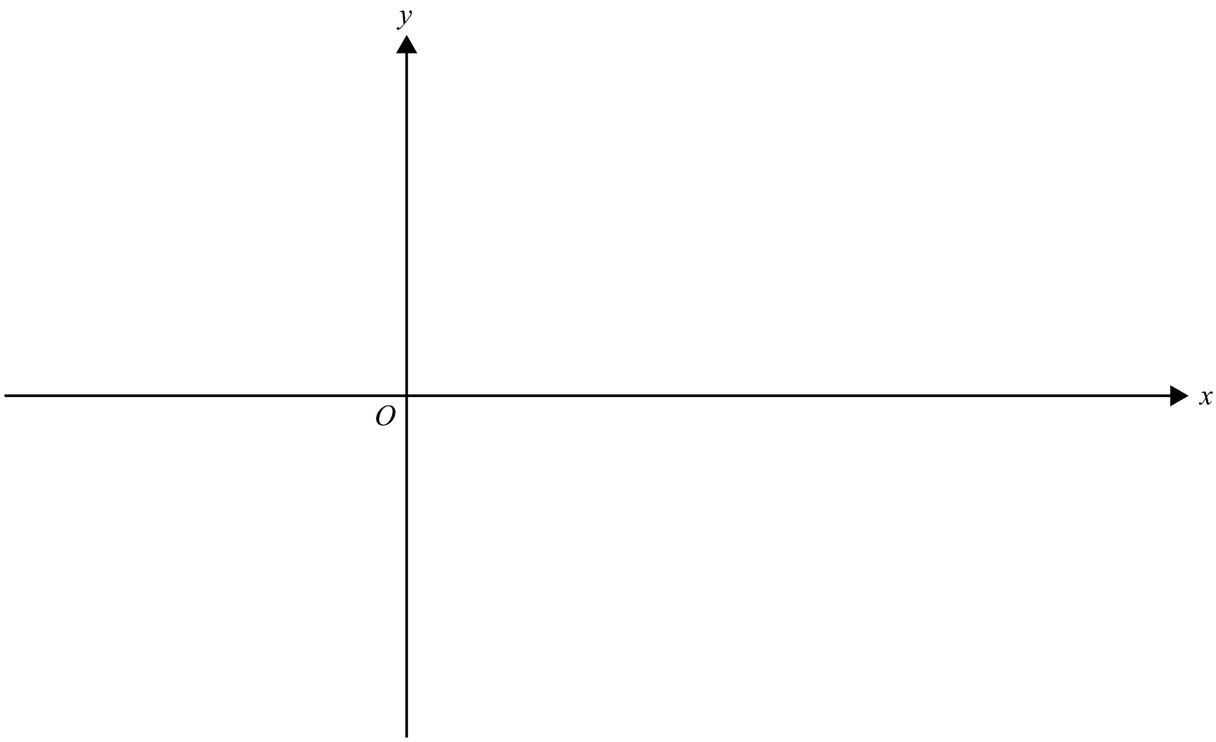
d. **i.** Find $h'(x)$.

ii. Solve the equation $h'(x) = 0$ for $x \in (-\pi, \pi) \cup (\pi, 3\pi)$. (Give exact values.)

1 + 2 = 3 marks

e. Sketch the graph of $y = h(x)$ on the axes below.

- Give the exact coordinates of any stationary points.
- Label each asymptote with its equation.
- Give the exact value of the y-intercept.



2 marks

Total 15 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

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Mathematical Methods and Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x} dx = \log_e |x| + c$

$\frac{d}{dx}\left(\sin(ax)\right) = a \cos(ax)$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a \sin(ax)$

$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a \sec^2(ax)$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET