VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2010

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Figures									Letter
Words									

MATHEMATICAL METHODS (CAS)

Written examination 2

Monday 8 November 2010

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 21 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The function with rule $f(x) = 4\tan\left(\frac{x}{3}\right)$ has period

A. $\frac{\pi}{3}$

B. 6π

C. 3

D. 3π

E. $\frac{2\pi}{3}$

Question 2

For $f(x) = x^3 + 2x$, the average rate of change with respect to x for the interval $[1, 5]$ is

A. 18
B. 20.5
C. 24
D. 32.5
E. 33

Question 3

The range of the function $f: R \to R, f(x) = |x^2 - 9| + 3$ is

A. $[3, \infty)$
B. R^+
C. $[-6, \infty)$
D. $R\backslash(-3, 3)$
E. R

Question 4

If $f(x) = \frac{1}{2}e^{3x}$ and $g(x) = \log_e(2x) + 3$ then $g(f(x))$ is equal to

A. $2x^3 + 3$
B. $e^{3x} + 3$
C. $e^{8x + 9}$
D. $3(x + 1)$
E. $\log_e(3x) + 3$

Question 5

For the system of simultaneous linear equations

$$x = 5$$
$$z + y = 10$$
$$z - y = 6$$

an equivalent matrix equation is

A. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$

C. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$

Question 6

A function g with domain R has the following properties.

- $g'(x) = x^2 - 2x$
- the graph of $g(x)$ passes through the point $(1, 0)$

$g(x)$ is equal to

A. $2x - 2$

B. $\frac{x^3}{3} - x^2$

C. $\frac{x^3}{3} - x^2 + \frac{2}{3}$

D. $x^2 - 2x + 2$

E. $3x^3 - x^2 - 1$

Question 7

The simultaneous linear equations $(m-1)x + 5y = 7$ and $3x + (m-3)y = 0.7m$ have infinitely many solutions for

A. $m \in R \setminus \{0, -2\}$

B. $m \in R \setminus \{0\}$

C. $m \in R \setminus \{6\}$

D. $m = 6$

E. $m = -2$

Question 8

The function f has rule $f(x) = 3\log_e(2x)$.

If $f(5x) = \log_e(y)$ then y is equal to

A. $30x$

B. $6x$

C. $125x^3$

D. $50x^3$

E. $1000x^3$

Question 9

The function $f: (-\infty, a] \to R$ with rule $f(x) = x^3 - 3x^2 + 3$ will have an inverse function provided

A. $a \leq 0$

B. $a \geq 2$

C. $a \geq 0$

D. $a \leq 2$

E. $a \leq 1$

Question 10

The average value of the function $f(x) = e^{2x}\cos(3x)$ for $0 \leq x \leq \pi$ is closest to

A. -82.5

B. 26.3

C. -26.3

D. -274.7

E. π

Question 11

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases}\cos(2x) & \text{if } \dfrac{3\pi}{4}<x<\dfrac{5\pi}{4}\\[2ex] 0 & \text{elsewhere}\end{cases}$$

The value of a such that $\Pr(X < a) = 0.25$ is closest to

A. 2.25

B. 2.75

C. 2.88

D. 3.06

E. 3.41

Question 12

A soccer player is practising her goal kicking. She has a probability of $\dfrac{3}{5}$ of scoring a goal with each attempt. She has 15 attempts.

The probability that the number of goals she scores is less than 7 is closest to

A. 0.0612

B. 0.0950

C. 0.1181

D. 0.2131

E. 0.7869

Question 13

The continuous random variable X has a normal distribution with mean 20 and standard deviation 6. The continuous random variable Z has the standard normal distribution.

The probability that Z is between –2 and 1 is equal to

A. $\Pr(18 < X < 21)$

B. $\Pr(14 < X < 32)$

C. $\Pr(14 < X < 26)$

D. $\Pr(8 < X < 32)$

E. $\Pr(X > 14) + \Pr(X < 26)$

Question 14

A bag contains four white balls and six black balls. Three balls are drawn from the bag without replacement. The probability that they are all black is

A. $\frac{1}{6}$

B. $\frac{27}{125}$

C. $\frac{24}{29}$

D. $\frac{3}{500}$

E. $\frac{8}{125}$

Question 15

The discrete random variable X has the following probability distribution.

X	0	1	2
$\Pr(X = x)$	a	b	0.4

If the mean of X is 1 then

A. $a = 0.3$ and $b = 0.1$

B. $a = 0.2$ and $b = 0.2$

C. $a = 0.4$ and $b = 0.2$

D. $a = 0.1$ and $b = 0.5$

E. $a = 0.1$ and $b = 0.3$

Question 16

The gradient of the function $f: R \to R, f(x) = \dfrac{5x}{x^2 + 3}$ is negative for

A. $-\sqrt{3} < x < \sqrt{3}$

B. $x > 3$

C. $x \in R$

D. $x < -\sqrt{3}$ and $x > \sqrt{3}$

E. $x < 0$

Question 17

The function f is differentiable for all $x \in R$ and satisfies the following conditions.

- $f'(x) < 0$ where $x < 2$
- $f'(x) = 0$ where $x = 2$
- $f'(x) = 0$ where $x = 4$
- $f'(x) > 0$ where $2 < x < 4$
- $f'(x) > 0$ where $x > 4$

Which one of the following is true?

A. The graph of f has a local maximum point where $x = 4$.

B. The graph of f has a stationary point of inflection where $x = 4$.

C. The graph of f has a local maximum point where $x = 2$.

D. The graph of f has a local minimum point where $x = 4$.

E. The graph of f has a stationary point of inflection where $x = 2$.

Question 18

For the function $f(x) = e^{|x|} - 1$, which of the following statements is true?

A. The function is increasing for all x.

B. The function has an asymptote at $y = -1$.

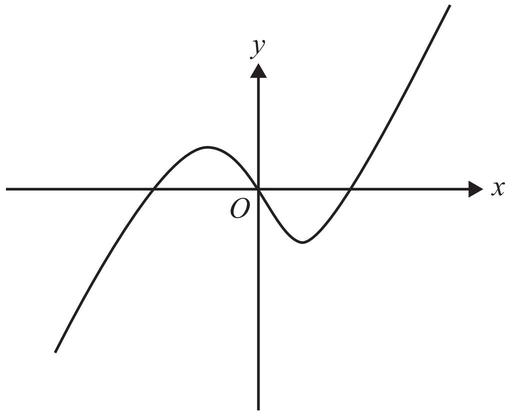
C. The function is not continuous at $x = 0$.

D. The function is not differentiable at $x = 0$.

E. The function has a stationary point at $x = 0$.

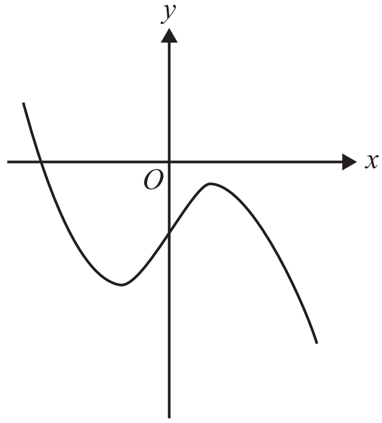
Question 19

The graph of the gradient function $y = f'(x)$ is shown below.

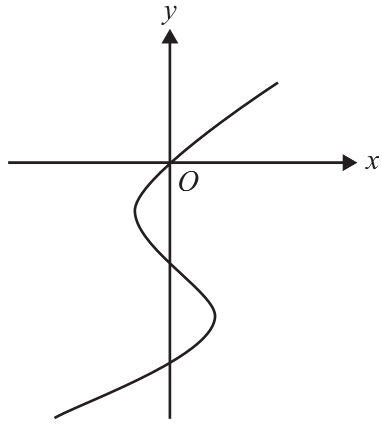


Which of the following could represent the graph of the function $f(x)$?

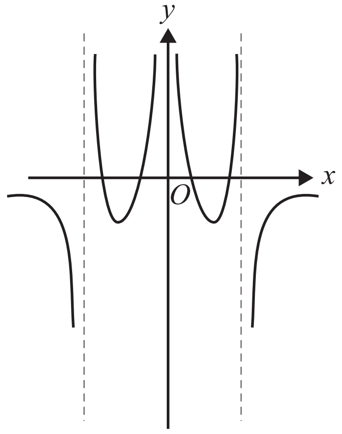
A.



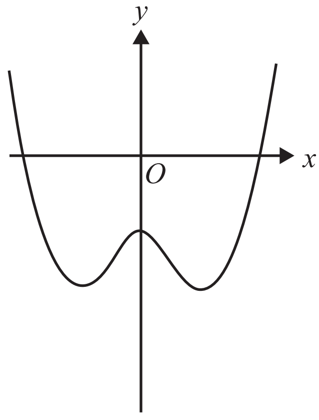
B.



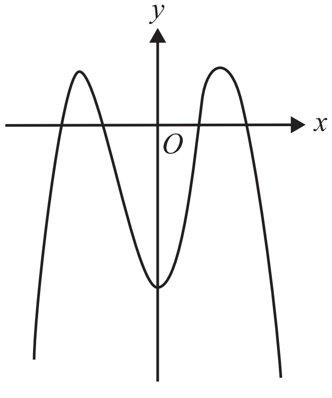
C.



D.



E.



Question 20

Let f be a differentiable function defined for all real x, where $f(x) \geq 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)dx = a$, then $2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)dx$ is equal to

A. $2a + 6$

B. $10a + 6$

C. $20a$

D. $40a$

E. $50a$

Question 21

Events A and B are mutually exclusive events of a sample space with

$$\Pr(A) = p \text{ and } \Pr(B) = q \text{ where } 0 < p < 1 \text{ and } 0 < q < 1.$$

$\Pr(A' \cap B')$ is equal to

A. $(1 - p)(1 - q)$

B. $1 - pq$

C. $1 - (p + q)$

D. $2 - p - q$

E. $1 - (p + q - pq)$

Question 22

Let f be a differentiable function defined for $x > 2$ such that

$$\int_3^{ab+2} f(x)dx = \int_3^{a+2} f(x)dx + \int_3^{b+2} f(x)dx \text{ where } a > 1 \text{ and } b > 1.$$

The rule for $f(x)$ is

A. $\sqrt{x-2}$

B. $\log_e(x-2)$

C. $\sqrt{2x-4}$

D. $\log_e|2x-4|$

E. $\dfrac{1}{x-2}$

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

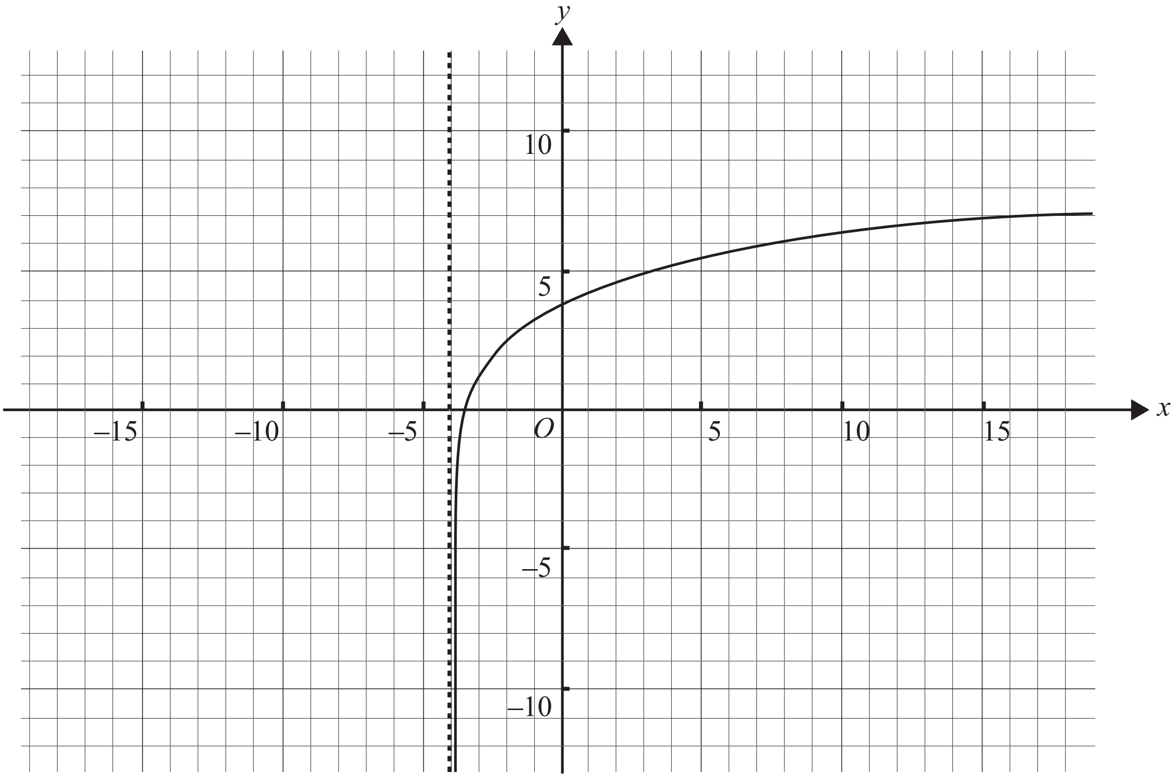
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. Part of the graph of the function $g: (-4, \infty) \rightarrow R$, $g(x) = 2\log_e(x + 4) + 1$ is shown on the axes below.



i. Find the rule and domain of g^{-1}, the inverse function of g.

ii. On the set of axes above sketch the graph of g^{-1}. Label the axes intercepts with their exact values.

iii. Find the values of x, correct to three decimal places, for which $g^{-1}(x) = g(x)$.

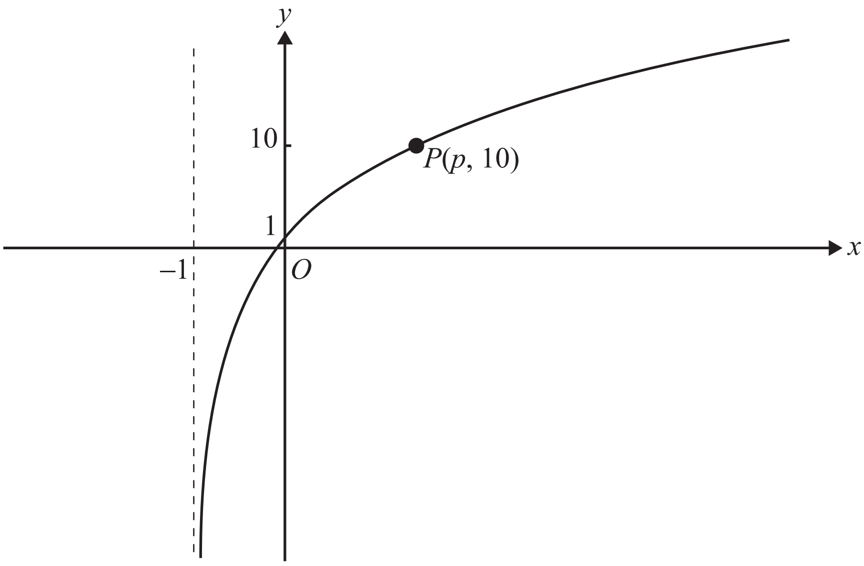
iv. Calculate the area enclosed by the graphs of g and g^{-1}. Give your answer correct to two decimal places.

3 + 3 + 2 + 2 = 10 marks

b. The diagram below shows part of the graph of the function with rule

$$f(x) = k \log_e(x + a) + c, \text{ where } k, a \text{ and } c \text{ are real constants.}$$

- The graph has a vertical asymptote with equation $x = -1$.
- The graph has a y-axis intercept at 1.
- The point P on the graph has coordinates $(p, 10)$, where p is another real constant.



i. State the value of a.

ii. Find the value of c.

iii. Show that $k = \dfrac{9}{\log_e(p+1)}$.

iv. Show that the gradient of the tangent to the graph of f at the point P is $\dfrac{9}{(p+1)\log_e(p+1)}$.

v. If the point (–1, 0) lies on the tangent referred to in **part b.iv.**, find the exact value of p.

1 + 1 + 2 + 1 + 2 = 7 marks

Total 17 marks

Question 2

Victoria Jones runs a small business making and selling statues of her cousin the adventurer Tasmania Jones.

The statues are made in a mould, then finished (smoothed and then hand-painted using a special gold paint) by Victoria herself. Victoria sends the statues **in order of completion** to an inspector, who classifies them as either ‘Superior’ or ‘Regular’, depending on the quality of their finish.

If a statue is Superior then the probability that the next statue completed is Superior is p.

If a statue is Regular then the probability that the next statue completed is Superior is $p - 0.2$.

On a particular day, Victoria knows that $p = 0.9$.

On that day

a. if the **first statue inspected is Superior**, find the probability that the third statue is Regular

2 marks

b. if the **first statue inspected is Superior**, find the probability that the next three statues are Superior

1 mark

c. find the steady state probability that any one of Victoria’s statues is Superior.

1 mark

On another day, Victoria finds that if the **first statue inspected is Superior** then the probability that the third statue is Superior is 0.7.

d. **i.** Show that the value of p on this day is 0.75.

On this day, a group of 3 consecutive statues is inspected. Victoria knows that the **first** statue of the 3 statues is **Regular**.

ii. Find the expected number of these 3 statues that will be Superior.

3 + 4 = 7 marks

Victoria hears that another company, Shoddy Ltd, is producing similar statues (also classified as Superior or Regular), but its statues are entirely made by machines, on a construction line. The quality of any one of Shoddy's statues is independent of the quality of any of the others on its construction line. The probability that any one of Shoddy's statues is Regular is 0.8.

Shoddy Ltd wants to ensure that the probability that it produces at least two Superior statues in a day's production run is at least 0.9.

e. Calculate the minimum number of statues that Shoddy would need to produce in a day to achieve this aim.

3 marks

Total 14 marks

Question 3

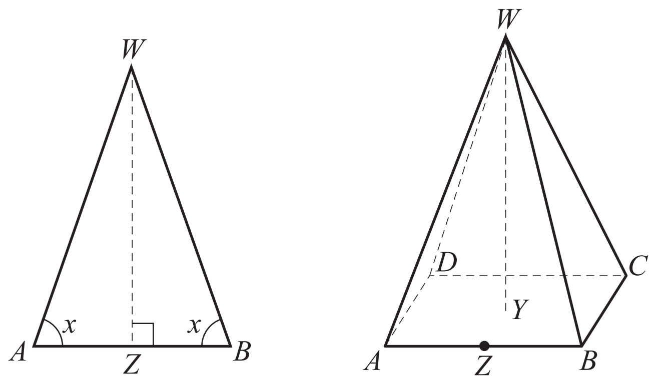
An ancient civilisation buried its kings and queens in tombs in the shape of a square-based pyramid, *WABCD*.

The kings and queens were each buried in a pyramid with $WA = WB = WC = WD = 10$ m.

Each of the isosceles triangle faces is congruent to each of the other triangular faces.

The base angle of each of these triangles is x, where $\frac{\pi}{4} < x < \frac{\pi}{2}$.

Pyramid *WABCD* and a face of the pyramid, *WAB*, are shown here.



Z is the midpoint of AB.

a. **i.** Find AB in terms of x.

ii. Find WZ in terms of x.

1 + 1 = 2 marks

b. Show that the total surface area (including the base), S m^2, of the pyramid, *WABCD*, is given by $S = 400(\cos^2(x) + \cos(x)\sin(x))$.

2 marks

c. Find WY, the height of the pyramid $WABCD$, in terms of x.

2 marks

d. The volume of any pyramid is given by the formula Volume = $\frac{1}{3}$ × area of base × vertical height.

Show that the volume, T m^3, of the pyramid $WABCD$ is $\frac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}$.

1 mark

Queen Hepzabah's pyramid was designed so that it had the **maximum possible volume**.

e. Find $\frac{dT}{dx}$ and **hence** find the exact volume of Queen Hepzabah's pyramid and the corresponding value of x.

4 marks

Queen Hepzabah's daughter, Queen Jepzibah, was also buried in a pyramid. It also had

$$WA = WB = WC = WD = 10 \text{ m.}$$

The volume of Jepzibah's pyramid is exactly one half of the volume of Queen Hepzabah's pyramid. The volume of Queen Jepzibah's pyramid is also given by the formula for T obtained in **part d.**

f. Find the possible values of x, for Jepzibah's pyramid, correct to two decimal places.

2 marks

Total 13 marks

Question 4

Consider the function $f: R \to R, f(x) = \frac{1}{27}(2x - 1)^3(6 - 3x) + 1$.

a. Find the x-coordinate of each of the stationary points of f and state the nature of each of these stationary points.

4 marks

In the following, f is the function $f: R \to R, f(x) = \frac{1}{27}(ax - 1)^3(b - 3x) + 1$ where a and b are real constants.

b. Write down, in terms of a and b, the possible values of x for which $(x, f(x))$ is a stationary point of f.

3 marks

c. For what value of a does f have no stationary points?

1 mark

d. Find a in terms of b if f has one stationary point.

2 marks

e. What is the maximum number of stationary points that f can have?

1 mark

f. Assume that there is a stationary point at (1, 1) and another stationary point (p, p) where $p \neq 1$.
Find the value of p.

3 marks

Total 14 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

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Mathematical Methods (CAS)
Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x} dx = \log_e |x| + c$

$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$

$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET