VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2023

Print exam correction:
Formula Sheet: page 2., Data analysis, probability and statistics table, 4th box, '+' has been added before X_n

STUDENT NUMBER | | | | | | | | | Letter | |

SPECIALIST MATHEMATICS
Written examination 1

Friday 3 November 2023

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 11 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

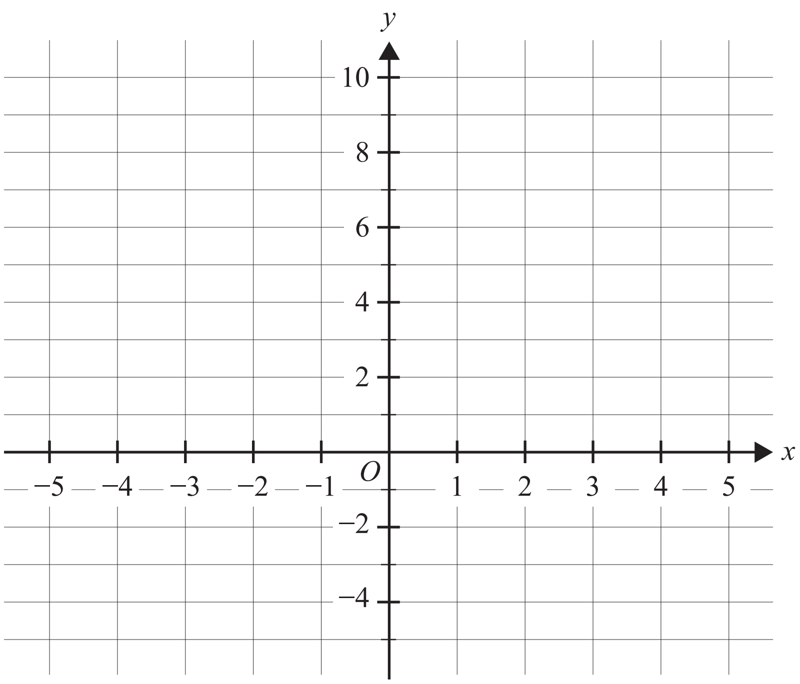
Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1 (4 marks)

Consider the function f with rule $f(x) = \dfrac{x^2 + x - 6}{x - 1}$.

a. Show that the rule for the function f can be written as $f(x) = x + 2 - \dfrac{4}{x-1}$. 1 mark

b. Sketch the graph of f on the axes below, labelling any asymptotes with their equations. 3 marks



Question 2 (3 marks)

Consider the complex number $z = (b - i)^3$, where $b \in R^+$.

Find b given that $\arg(z) = -\dfrac{\pi}{2}$.

Question 3 (3 marks)

A particle moves along a straight line. When the particle is x m from a fixed point O, its velocity, v m s^{-1}, is given by

$$v = \frac{3x + 2}{2x - 1}, \text{ where } x \geq 1.$$

a. Find the acceleration of the particle, in m s^{-2}, when $x = 2$. 2 marks

b. Find the value that the velocity of the particle approaches as x becomes very large. 1 mark

Question 4 (3 marks)

Consider the relation $x \arcsin(y^2) = \pi$.

Use implicit differentiation to find $\frac{dy}{dx}$ at the point $\left(6, \frac{1}{\sqrt{2}}\right)$.

Give your answer in the form $-\frac{\pi\sqrt{a}}{b}$, where $a, b \in Z^+$.

Question 5 (3 marks)

Evaluate $\int_1^2 x^2 \log_e(x)\,dx$.

Question 6 (4 marks)

Josie travels from home to work in the city. She drives a car to a train station, waits, and then rides on a train to the city. The time, X_c minutes, taken to drive to the station is normally distributed with a mean of 20 minutes ($\mu_c = 20$) and standard deviation of 6 minutes ($\sigma_c = 6$). The waiting time, X_w minutes, for a train is normally distributed with a mean of 8 minutes ($\mu_w = 8$) and standard deviation of $\sqrt{3}$ minutes $\left(\sigma_w = \sqrt{3}\right)$. The time, X_t minutes, taken to ride on a train to the city is also normally distributed with a mean of 12 minutes ($\mu_t = 12$) and standard deviation of 5 minutes ($\sigma_t = 5$). The three times are independent of each other.

a. Find the mean and standard deviation of the total time, in minutes, it takes for Josie to travel from home to the city. 2 marks

b. Josie's waiting time for a train on each work day is independent of her waiting time for a train on any other work day. The probability that, for 12 randomly chosen work days, Josie's average waiting time is between 7 minutes 45 seconds and 8 minutes 30 seconds is equivalent to $\Pr(a < Z < b)$, where $Z \sim N(0, 1)$ and a and b are real numbers.

Find the values of a and b. 2 marks

Question 7 (4 marks)

The curve defined by the parametric equations

$$x = \frac{t^2}{4} - 1, \ \ y = \sqrt{3}\,t, \text{ where } 0 \le t \le 2,$$

is rotated about the x-axis to form an open hollow surface of revolution.

Find the surface area of the surface of revolution.

Give your answer in the form $\pi\left(\frac{a\sqrt{b}}{c} - d\right)$, where a, b, c and $d \in Z^+$.

DO NOT WRITE IN THIS AREA

Question 8 (4 marks)

A function f has the rule $f(x) = x\,e^{2x}$.

Use mathematical induction to prove that $f^{(n)}(x) = \left(2^n x + n\,2^{n-1}\right)e^{2x}$ for $n \in Z^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$. That is, $f(x)$ has been differentiated n times.

DO NOT WRITE IN THIS AREA

Question 9 (6 marks)

A plane contains the points $A(1, 3, -2)$, $B(-1, -2, 4)$ and $C(a, -1, 5)$, where a is a real constant. The plane has a y-axis intercept of 2 at the point D.

a. Write down the coordinates of point D. 1 mark

b. Show that $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are $-2\underset{\sim}{i} - 5\underset{\sim}{j} + 6\underset{\sim}{k}$ and $-\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, respectively. 1 mark

c. Hence find the equation of the plane in Cartesian form. 2 marks

d. Find a. 1 mark

e. $\overline{AB}$ and $\overline{AD}$ are adjacent sides of a parallelogram. Find the area of this parallelogram. 1 mark

Question 10 (6 marks)

The position vector of a particle at time t seconds is given by

$$\underset{\sim}{r}(t) = \left(5 - 6\sin^2(t)\right)\underset{\sim}{i} + \left(1 + 6\sin(t)\cos(t)\right)\underset{\sim}{j}, \text{ where } t \geq 0.$$

a. Write $5 - 6\sin^2(t)$ in the form $\alpha + \beta\cos(2t)$, where $\alpha, \beta \in Z^+$. 1 mark

b. Show that the Cartesian equation of the path of the particle is $(x-2)^2 + (y-1)^2 = 9$. 2 marks

c. The particle is at point A when $t = 0$ and at point B when $t = a$, where a is a positive real constant.

If the distance travelled along the curve from A to B is $\frac{3\pi}{4}$, find a. 1 mark

Question 10 – continued

d. Find all values of t for which the position vector of the particle, $\underset{\sim}{r}(t)$, is perpendicular to its velocity vector, $\dot{\underset{\sim}{r}}(t)$. 2 marks

END OF QUESTION AND ANSWER BOOK

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SPECIALIST MATHEMATICS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference. A question and answer book is provided with this formula sheet.

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Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$	sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
volume of a pyramid	$\frac{1}{3}Ah$	cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\text{cis}(\theta)$	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	
$-\pi < \text{Arg}(z) \le \pi$	$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$	
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$	de Moivre's theorem	$z^n = r^n \text{cis}(n\theta)$

Data analysis, probability and statistics

for independent random variables $X_1, X_2 \ldots X_n$	$\text{E}(aX_1 + b) = a\,\text{E}(X_1) + b$ $\text{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1\text{E}(X_1) + a_2\text{E}(X_2) + \ldots + a_n\text{E}(X_n)$	
	$\text{Var}(aX_1 + b) = a^2\text{Var}(X_1)$ $\text{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2) + \ldots + a_n^2\text{Var}(X_n)$	
for independent identically distributed variables $X_1, X_2 \ldots X_n$	$\text{E}(X_1 + X_2 + \ldots + X_n) = n\mu$	
	$\text{Var}(X_1 + X_2 + \ldots + X_n) = n\sigma^2$	
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$	
distribution of sample mean $\bar{X}$	mean	$\text{E}(\bar{X}) = \mu$
	variance	$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$
$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$
$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$
$\frac{d}{dx}(\tan(ax)) = a\sec^2(ax)$
$\frac{d}{dx}(\cot(ax)) = -a\,\text{cosec}^2(ax)$
$\frac{d}{dx}(\sec(ax)) = a\sec(ax)\tan(ax)$
$\frac{d}{dx}(\text{cosec}(ax)) = -a\,\text{cosec}(ax)\cot(ax)$
$\frac{d}{dx}(\sin^{-1}(ax)) = \frac{a}{\sqrt{1-(ax)^2}}$
$\frac{d}{dx}(\cos^{-1}(ax)) = \frac{-a}{\sqrt{1-(ax)^2}}$
$\frac{d}{dx}(\tan^{-1}(ax)) = \frac{a}{1+(ax)^2}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, \quad n \neq -1$
$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\int \frac{1}{x} dx = \log_e \lvert x \rvert + c$
$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\int \operatorname{cosec}^2(ax)dx = -\frac{1}{a}\cot(ax) + c$
$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + c$
$\int \operatorname{cosec}(ax)\cot(ax)\,dx = -\frac{1}{a}\operatorname{cosec}(ax) + c$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, \quad n \neq -1$
$\int \frac{1}{ax+b} dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$

Calculus – continued

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
integration by parts	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
Euler's method	If $\frac{dy}{dx} = f(x, y)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f(x_n, y_n)$.
arc length parametric	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area Cartesian about x-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
surface area Cartesian about y-axis	$\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
surface area parametric about x-axis	$\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area parametric about y-axis	$\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	
constant acceleration formulas	$v = u + at$	$s = ut + \frac{1}{2}at^2$
	$v^2 = u^2 + 2as$	$s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
for $\underset{\sim}{r}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$	vector scalar product $\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = \lvert\underset{\sim}{r}_1\rvert\lvert\underset{\sim}{r}_2\rvert\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
	vector cross product $\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$
vector equation of a line	$\underset{\sim}{r}(t) = \underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_1 + x_2t)\underset{\sim}{i} + (y_1 + y_2t)\underset{\sim}{j} + (z_1 + z_2t)\underset{\sim}{k}$
parametric equation of a line	$x(t) = x_1 + x_2t \quad y(t) = y_1 + y_2t \quad z(t) = z_1 + z_2t$
vector equation of a plane	$\underset{\sim}{r}(s, t) = \underset{\sim}{r}_0 + s\underset{\sim}{r}_1 + t\underset{\sim}{r}_2$ $= (x_0 + x_1s + x_2t)\underset{\sim}{i} + (y_0 + y_1s + y_2t)\underset{\sim}{j} + (z_0 + z_1s + z_2t)\underset{\sim}{k}$
parametric equation of a plane	$x(s, t) = x_0 + x_1s + x_2t,\ y(s, t) = y_0 + y_1s + y_2t,\ z(s, t) = z_0 + z_1s + z_2t$
Cartesian equation of a plane	$ax + by + cz = d$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

END OF FORMULA SHEET