STUDENT NUMBER

Figures

Words

MATHEMATICAL METHODS

Written examination 2

Monday 10 November 2008

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

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• Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved graphics calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.

• Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

• Question and answer book of 24 pages with a detachable sheet of miscellaneous formulas in the centrefold.
• Answer sheet for multiple-choice questions.

Instructions

• Detach the formula sheet from the centre of this book during reading time.
• Write your student number in the space provided above on this page.
• Check that your name and student number as printed on your answer sheet for multiple-choice questions are correct, and sign your name in the space provided to verify this.
• All written responses must be in English.

At the end of the examination

• Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION 1

Instructions for Section 1
Answer all questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is correct for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1
The derivative of \( \frac{\log_e(2x)}{2x} \) with respect to \( x \) is
A. \( \frac{1}{x} \)
B. \( \frac{1 - 2 \log_e(2x)}{4x^2} \)
C. \( \frac{\log_e(2x)}{2x^2} \)
D. \( \frac{1 - \log_e(x)}{x^2} \)
E. \( \frac{1 - \log_e(2x)}{2x^2} \)

Question 2

[Graph image]

The rule of the function whose graph is shown is
A. \( y = |x| - 4 \)
B. \( y = |x - 2| + 2 \)
C. \( y = |x + 2| - 2 \)
D. \( y = |2 - x| - 2 \)
E. \( y = |2 + x| + 2 \)
Question 3
The solution of the equation \(3 \log_e(2x - 3) - 6 = 0\) is
A. 1.5
B. 2.5
C. \(\frac{1}{2}(\log_e(2) + 3)\)
D. \(\frac{1}{2}(e^2 + 3)\)
E. \(\frac{1}{2}(e^6 + 9)\)

Question 4
If \(\int_1^3 f(x)\,dx = 5\), then \(\int_1^3 (2f(x) - 3)\,dx\) is equal to
A. 4
B. 5
C. 7
D. 10
E. 16

Question 5
Let \(X\) be a discrete random variable with a binomial distribution. The mean of \(X\) is 1.2 and the variance of \(X\) is 0.72.
The values of \(n\) (the number of independent trials) and \(p\) (the probability of success in each trial) are
A. \(n = 4, p = 0.3\)
B. \(n = 3, p = 0.6\)
C. \(n = 2, p = 0.6\)
D. \(n = 2, p = 0.4\)
E. \(n = 3, p = 0.4\)

Question 6
\(\int \left( e^{3(x-2)} + \frac{2}{2-x} \right) \,dx\) is equal to
A. \(\frac{1}{3} e^{3(x-2)} - 2 \log_e|x-2| + c\)
B. \(\frac{1}{3} e^{3(x-2)} + 2 \log_e|2-x| + c\)
C. \(3e^{3(x-2)} + 2 \log_e|2-x| + c\)
D. \(-2 \log_e|x-2| + 3e^{3(x-2)} + c\)
E. \(3e^{3(x-2)} + \frac{2}{(2-x)^2} + c\)
Question 7
The inverse of the function \( f: \mathbb{R}^+ \rightarrow \mathbb{R}, \ f(x) = \frac{1}{\sqrt{x}} - 3 \) is

A. \( f^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R} \) \( f^{-1}(x) = (x + 3)^2 \)

B. \( f^{-1}: \mathbb{R}^+ \rightarrow \mathbb{R} \) \( f^{-1}(x) = \frac{1}{x^2} + 3 \)

C. \( f^{-1}: (3, \infty) \rightarrow \mathbb{R} \) \( f^{-1}(x) = \frac{-1}{(x - 3)^2} \)

D. \( f^{-1}: (-3, \infty) \rightarrow \mathbb{R} \) \( f^{-1}(x) = \frac{1}{(x + 3)^2} \)

E. \( f^{-1}: (-3, \infty) \rightarrow \mathbb{R} \) \( f^{-1}(x) = -\frac{1}{x^2} - 3 \)

Question 8
The graph of the function \( f: D \rightarrow \mathbb{R}, \ f(x) = \frac{x - 3}{2 - x} \), where \( D \) is the maximal domain, has asymptotes

A. \( x = 3, \ y = 2 \)

B. \( x = -2, \ y = 1 \)

C. \( x = 1, \ y = -1 \)

D. \( x = 2, \ y = -1 \)

E. \( x = 2, \ y = 1 \)

Question 9
\[ \int_{1}^{3} \left( 6x^2 + \frac{3a}{x^2} \right) dx, \ \text{where} \ a \ \text{is a real number, is equal to} \]

A. \( 52 - 2a \)

B. \( 24 + 4a \)

C. \( 24 + 2a \)

D. \( 52 + 2a \)

E. \( 52 + \frac{26a}{27} \)

Question 10
The range of the function \( f: \left[ \frac{\pi}{8}, \frac{\pi}{3} \right) \rightarrow \mathbb{R}, \ f(x) = 2 \sin(2x) \) is

A. \( \left( \sqrt{2}, \sqrt{3} \right) \)

B. \( \left[ \sqrt{2}, 2 \right) \)

C. \( \left[ \sqrt{2}, 2 \right] \)

D. \( \left( \sqrt{2}, \sqrt{3} \right) \)

E. \( \left[ \sqrt{2}, \sqrt{3} \right] \)
Question 11

The probability density function for the continuous random variable $X$ is given by

$$f(x) = \begin{cases} |1-x| & \text{if } 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

The probability that $X < 1.5$ is equal to

A. 0.125  
B. 0.375  
C. 0.5  
D. 0.625  
E. 0.75

Question 12

If $f(x) = \frac{\sin(2\pi x)}{g(x)}$ and $g(x) \neq 0$ then $f'(x)$ is equal to

A. $\frac{\cos(2\pi x)}{g'(x)}$  
B. $\frac{2\pi g(x)\cos(2\pi x) - \sin(2\pi x)g'(x)}{[g(x)]^2}$  
C. $\frac{2\pi g(x)\sin(2\pi x) - \cos(2\pi x)g'(x)}{[g(x)]^2}$  
D. $\frac{2\pi g(x)(\cos(2\pi x) - \sin(2\pi x)g'(x))}{[g(x)]^2}$  
E. $\frac{2\pi \cos(2\pi x)}{g'(x)}$

Question 13

According to a survey, 30% of employed women have never been married.

If 10 employed women are selected at random, the probability (correct to four decimal places) that at least 7 have not been married is

A. 0.0016  
B. 0.0090  
C. 0.0106  
D. 0.9894  
E. 0.9984

Question 14

The minimum number of times that a fair coin can be tossed so that the probability of obtaining a head on each trial is less than 0.0005 is

A. 8  
B. 9  
C. 10  
D. 11  
E. 12
Question 15
The sample space when a fair die is rolled is \{1, 2, 3, 4, 5, 6\}, with each outcome being equally likely.
For which of the following pairs of events are the events independent?
A. \{1, 2, 3\} and \{1, 2\}
B. \{1, 2\} and \{3, 4\}
C. \{1, 3, 5\} and \{1, 4, 6\}
D. \{1, 2\} and \{1, 3, 4, 6\}
E. \{1, 2\} and \{2, 4, 6\}

Question 16
Water is being poured into a long cylindrical storage tank of radius 2 metres, with its circular base on the ground, at a rate of 2 cubic metres per second.

The rate of change of the depth of the water, in metres per second, in the tank is
A. \(\frac{1}{8\pi}\)
B. \(\frac{1}{4\pi}\)
C. \(\frac{1}{2\pi}\)
D. 2\(\pi\)
E. 8\(\pi\)

Question 17
The graph of \(f(x) = e^{2x} - 2\) intersects the graph of \(g(x) = e^x\) where
A. \(x = -1\)
B. \(x = \log_{e}(2)\)
C. \(x = 2\)
D. \(x = \frac{1 + \sqrt{7}}{2}\)
E. \(x = \log_{e}\left(\frac{1 + \sqrt{7}}{2}\right)\)
Question 18

Let \( f: \left[ 0, \frac{\pi}{2} \right] \rightarrow \mathbb{R}, \ f(x) = \sin(4x) + 1 \). The graph of \( f \) is transformed by a reflection in the \( x \)-axis followed by a dilation of factor 4 from the \( y \)-axis.

The resulting graph is defined by

A. \( g: \left[ 0, \frac{\pi}{2} \right] \rightarrow \mathbb{R}, \ g(x) = -1 - 4 \sin(4x) \)

B. \( g: [0, 2\pi] \rightarrow \mathbb{R}, \ g(x) = -1 - \sin(16x) \)

C. \( g: \left[ 0, \frac{\pi}{2} \right] \rightarrow \mathbb{R}, \ g(x) = 1 - \sin(x) \)

D. \( g: [0, 2\pi] \rightarrow \mathbb{R}, \ g(x) = 1 - \sin(4x) \)

E. \( g: [0, 2\pi] \rightarrow \mathbb{R}, \ g(x) = -1 - \sin(x) \)
Question 19
The graph of a function $f$ is shown below.

The graph of an antiderivative of $f$ could be

A. 

B. 

C. 

D. 

E.
Question 20
The function \( f: \mathbb{B} \to \mathbb{R} \) with rule \( f(x) = 4x^3 + 3x^2 + 1 \) will have an inverse function for

A. \( B = \mathbb{R} \)
B. \( B = \left( \frac{1}{2}, \infty \right) \)
C. \( B = \left( -\infty, \frac{1}{2} \right] \)
D. \( B = \left( -\infty, \frac{1}{2} \right) \)
E. \( B = \left[ -\frac{1}{2}, \infty \right) \)

Question 21
The graph of \( y = x^3 - 12x \) has turning points where \( x = 2 \) and \( x = -2 \).
The graph of \( y = |x^3 - 12x| \) has a positive gradient for

A. \( x \in \mathbb{R} \)
B. \( x \in \{ x : x < -2 \} \cup \{ x : x > 2 \} \)
C. \( x \in \{ x : x < -2\sqrt{3} \} \cup \{ x : x > 2\sqrt{3} \} \)
D. \( x \in \{ x : -2\sqrt{3} < x < -2 \} \cup \{ x : 0 < x < 2 \} \cup \{ x : x > 2\sqrt{3} \} \)
E. \( x \in \{ x : -2 < x < 0 \} \cup \{ x : 2 < x < 2\sqrt{3} \} \cup \{ x : x > 2\sqrt{3} \} \)

Question 22
The graph of the function \( f \) with domain \([0, 6]\) is shown below.

![Graph of the function](image)

Which one of the following is not true?

A. The function is not continuous at \( x = 2 \) and \( x = 4 \).
B. The function exists for all values of \( x \) between 0 and 6.
C. \( f(x) = 0 \) for \( x = 2 \) and \( x = 5 \).
D. The function is positive for \( x \in [0, 5] \).
E. The gradient of the function is not defined at \( x = 4 \).
SECTION 2

Instructions for Section 2

Answer all questions in the spaces provided.
A decimal approximation will not be accepted if an exact answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Where an instruction to use calculus is stated for a question, you must show an appropriate derivative or anti-derivative.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1
Sharelle is the goal shooter for her netball team. During her matches, she has many attempts at scoring a goal.
Assume that each attempt at scoring a goal is independent of any other attempt. In the long term, her scoring rate has been shown to be 80% (that is, 8 out of 10 attempts to score a goal are successful).

a. i. What is the probability, correct to four decimal places, that her first 8 attempts at scoring a goal in a match are successful?

ii. What is the probability, correct to four decimal places, that exactly 6 of her first 8 attempts at scoring a goal in a match are successful?

iii. What is the probability, correct to three decimal places, that her first 4 attempts at scoring a goal are successful, given that exactly 6 of her first 8 attempts at scoring a goal in a match are successful?

1 + 2 + 2 = 5 marks
Assume instead that the success of an attempt to score a goal depends only on the success or otherwise of her previous attempt at scoring a goal.

If an attempt at scoring a goal in a match is successful, then the probability that her next attempt at scoring a goal in the match is successful is 0.84. However, if an attempt at scoring a goal in a match is unsuccessful, then the probability that her next attempt at scoring a goal in the match is successful is 0.64.

**Her first attempt at scoring a goal in a match is successful.**

b. i. What is the probability, correct to four decimal places, that her next 7 attempts at scoring a goal in the match will be successful?

ii. What is the probability, correct to four decimal places, that exactly 2 of her next 3 attempts at scoring a goal in the match will be successful?
The time in hours that Sharelle spends training each day is a continuous random variable with probability density function given by

\[ f(x) = \begin{cases} \frac{1}{64}(6-x)(x-2)(x+2) & \text{if } 2 \leq x \leq 6 \\ 0 & \text{elsewhere} \end{cases} \]

c.  

i. Sketch the probability density function, and label the local maximum with its coordinates, correct to two decimal places.

ii. What is the probability, correct to four decimal places, that Sharelle spends less than 3 hours training on a particular day?

iii. What is the mean time (in hours), correct to four decimal places, that she spends training each day?

2 + 2 + 2 = 6 marks
Total 15 marks
Working space
**Question 2**

The diagram below shows part of the graph of the function \( f: \mathbb{R}^+ \rightarrow \mathbb{R}, \ f(x) = \frac{7}{x}. \)

The line segment \( CA \) is drawn from the point \( C(1, f(1)) \) to the point \( A(a, f(a)) \) where \( a > 1. \)

**a.**

i. Calculate the gradient of \( CA \) in terms of \( a. \)

ii. At what value of \( x \) between 1 and \( a \) does the tangent to the graph of \( f \) have the same gradient as \( CA? \)

1 + 2 = 3 marks
b. i. Calculate \( \int_{1}^{e} f(x) \, dx \).

ii. Let \( b \) be a positive real number less than one. Find the exact value of \( b \) such that \( \int_{b}^{1} f(x) \, dx \) is equal to 7.

2 + 2 = 4 marks

c. i. Express the area of the region bounded by the line segment \( CA \), the \( x \)-axis, the line \( x = 1 \) and the line \( x = a \) in terms of \( a \).

ii. For what exact value of \( a \) does this area equal 7?
iii. Using the value for \( a \) determined in c.ii., explain in words, without evaluating the integral, why \( \int_{1}^{a} f(x) \, dx < 7 \).

Use this result to explain why \( a < e \).
Working space
Question 3
Tasmania Jones is in the jungle, digging for gold. He finds the gold at $X$ which is 3 km from a point $A$. Point $A$ is on a straight beach.
Tasmania’s camp is at $Y$ which is a distance of 3 km from a point $B$. Point $B$ is also on the straight beach. $AB = 18$ km and $AM = NB = x$ km and $AX = BY = 3$ km.

While he is digging up the gold, Tasmania is bitten by a snake which injects toxin into his blood. After he is bitten, the concentration of the toxin in his bloodstream increases over time according to the equation

$$y = 50 \log_e(1 + 2t)$$

where $y$ is the concentration, and $t$ is the time in hours after the snake bites him.
The toxin will kill him if its concentration reaches 100.

a. Find the time, to the nearest minute, that Tasmania has to find an antidote (that is, a cure for the toxin).

b. Show that he will not get the antidote in time if he runs directly to his camp through the jungle.

---

2 marks

Tasmania has an antidote to the toxin at his camp. He can run through the jungle at 5 km/h and he can run along the beach at 13 km/h.

b. Show that he will not get the antidote in time if he runs directly to his camp through the jungle.

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1 mark
In order to get the antidote, Tasmania runs through the jungle to \( M \) on the beach, runs along the beach to \( N \) and then runs through the jungle to the camp at \( Y \). \( M \) is \( x \) km from \( A \) and \( N \) is \( x \) km from \( B \). (See diagram.)

c. Show that the time taken to reach the camp, \( T \) hours, is given by

\[
T = 2 \left( \frac{\sqrt{9 + x^2}}{5} + \frac{9 - x}{13} \right)
\]

d. Use calculus to find the value of \( x \) which allows Tasmania to get to his camp in the minimum time.

e. Show that he gets to his camp in time to get the antidote.
At his camp, Tasmania Jones takes a capsule containing 16 units of antidote to the toxin. After taking the capsule the quantity of antidote in his body decreases over time.

At exactly the same time on successive days, he takes another capsule containing 16 units of antidote and again the quantity of antidote decreases in his body.

The graph of the quantity of antidote $z$ units in his body at time $d$ days after taking the first capsule looks like this. Each section of curve has exactly the same shape as curve $AB$.

The equation of the curve $AB$ is $z = \frac{16}{d+1}$

f. Write down the coordinates of the points $A$ and $C$.

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

2 marks

g. Find the equation of the curve $CD$.

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2 marks
Tasmania will no longer be affected by the snake toxin when he first has 50 units of the antidote in his body.

h. Assuming he takes a capsule at the same time each day, on how many days does he need to take a capsule so that he will no longer be affected by the snake toxin?
Question 4
The graph of \( f: (-\pi, \pi) \cup (\pi, 3\pi) \to \mathbb{R}, \ f(x) = \tan\left(\frac{x}{2}\right) \) is shown below.

a. 
   i. Use calculus to find \( f'(\frac{\pi}{2}) \).

   ___________________________________________________________________________

   ___________________________________________________________________________

   ___________________________________________________________________________

   ii. Find the equation of the normal to the graph of \( y = f(x) \) at the point where \( x = \frac{\pi}{2} \).

   ___________________________________________________________________________

   ___________________________________________________________________________

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   iii. Sketch the graph of this normal on the axes above. Give the exact axis intercepts.

   ___________________________________________________________________________

   ___________________________________________________________________________

   ___________________________________________________________________________

\[ 2 + 2 + 3 = 7 \text{ marks} \]
b. Find the exact values of \( x \in (-\pi, \pi) \cup (\pi, 3\pi) \) such that \( f'(x) = f'(\frac{\pi}{2}) \).

Let \( g(x) = f(x - a) \).

c. Find the exact value of \( a \in (-1, 1) \) such that \( g(1) = 1 \).

Let \( h: (-\pi, \pi) \cup (\pi, 3\pi) \to \mathbb{R}, h(x) = \sin\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right) + 2 \).

d. i. Use calculus to find \( h'(x) \).

ii. Solve the equation \( h'(x) = 0 \) for \( x \in (-\pi, \pi) \cup (\pi, 3\pi) \). (Give exact values.)

1 + 2 = 3 marks
e. Sketch the graph of \( y = h(x) \) on the axes below.
   - Give the exact coordinates of any stationary points.
   - Label each asymptote with its equation.
   - Give the exact value of the \( y \)-intercept.
MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

| Detach this formula sheet during reading time. |
| This formula sheet is provided for your reference. |
Mathematical Methods and Mathematical Methods (CAS)

Formulas

Mensuration

area of a trapezium: \[ \frac{1}{2}(a + b)h \]

volume of a pyramid: \[ \frac{1}{3}Ah \]

curved surface area of a cylinder: \[ 2\pi rh \]

volume of a cylinder: \[ \pi r^2h \]

area of a triangle: \[ \frac{1}{2}bc \sin A \]

Calculus

\[ \frac{d}{dx}(x^n) = nx^{n-1} \]

\[ \frac{d}{dx}(e^{ax}) = ae^{ax} \]

\[ \frac{d}{dx}(\log_e(x)) = \frac{1}{x} \]

\[ \frac{d}{dx}(\sin(ax)) = a \cos(ax) \]

\[ \frac{d}{dx}(\cos(ax)) = -a \sin(ax) \]

\[ \frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a \sec^2(ax) \]

product rule: \[ \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

quotient rule: \[ \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

chain rule: \[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \]

approximation: \[ f(x + h) \approx f(x) + hf'(x) \]

Probability

\[ \Pr(A) = 1 - \Pr(A') \]

\[ \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \]

\[ \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \]

mean: \[ \mu = \mathbb{E}(X) \]

variance: \[ \sigma^2 = \mathbb{E}((X - \mu)^2) = \mathbb{E}(X^2) - \mu^2 \]

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<th>mean</th>
<th>variance</th>
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<tr>
<td>discrete [ \Pr(X = x) = p(x) ]</td>
<td>[ \mu = \sum x , p(x) ]</td>
<td>[ \sigma^2 = \sum (x - \mu)^2 , p(x) ]</td>
</tr>
<tr>
<td>continuous [ \Pr(a &lt; X &lt; b) = \int_a^b f(x)dx ]</td>
<td>[ \mu = \int_{-\infty}^{\infty} x , f(x)dx ]</td>
<td>[ \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 , f(x)dx ]</td>
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