SPECIALIST MATHEMATICS

Written examination 1

Friday 6 November 2015

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

<table>
<thead>
<tr>
<th>Number of questions</th>
<th>Number of questions to be answered</th>
<th>Number of marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>9</td>
<td>40</td>
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</tbody>
</table>

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are not permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 10 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your student number in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer all questions in the spaces provided.

Unless otherwise specified, an exact answer is required to a question.

In questions where more than one mark is available, appropriate working must be shown.

Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Take the acceleration due to gravity to have magnitude $g$ m/s$^2$, where $g = 9.8$.

Question 1 (3 marks)

Consider the rhombus $OABC$ shown below, where $\overrightarrow{OA} = ai$ and $\overrightarrow{OC} = i + j + k$, and $a$ is a positive real constant.

\[ \begin{array}{c}
 C \\
 \downarrow \\
 B \\
 \downarrow \\
 A \\
 \downarrow \\
 O 
\end{array} \]

a. Find $a$. 1 mark

b. Show that the diagonals of the rhombus $OABC$ are perpendicular. 2 marks
Question 2 (4 marks)
A 20 kg parcel sits on the floor of a lift.

a. The lift is accelerating upwards at 1.2 ms$^{-2}$.
   Find the reaction force of the lift floor on the parcel in newtons.  
   \[ \text{2 marks} \]

b. Find the acceleration of the lift downwards in ms$^{-2}$ so that the reaction of the lift floor on the parcel is 166 N.  
   \[ \text{2 marks} \]

Question 3 (4 marks)
The velocity of a particle at time $t$ seconds is given by \( \vec{v}(t) = (4t - 3)\hat{i} + 2t\hat{j} - 5\hat{k} \), where components are measured in metres per second.

Find the distance of the particle from the origin in metres when $t = 2$, given that $\vec{r}(0) = \hat{i} - 2\hat{k}$.  

\[ \]
Question 4 (4 marks)
a. Find all solutions of $z^3 = 8i, z \in C$ in cartesian form.  

b. Find all solutions of $(z - 2i)^3 = 8i, z \in C$ in cartesian form.

Question 5 (3 marks)
Find the volume generated when the region bounded by the graph of $y = 2x^2 - 3$, the line $y = 5$ and the $y$-axis is rotated about the $y$-axis.
Question 6 (4 marks)
The acceleration $a$ ms$^{-2}$ of a body moving in a straight line in terms of the velocity $v$ ms$^{-1}$ is given by $a = 4v^2$.

Given that $v = e$ when $x = 1$, where $x$ is the displacement of the body in metres, find the velocity of the body when $x = 2$. 
Question 7 (5 marks)

a. Solve \( \sin(2x) = \sin(x), \, x \in [0, 2\pi] \).  

b. Find \( \{ x : \csc(2x) < \csc(x), \, x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \} \).
Question 8 (7 marks)

a. Show that \( \int \tan(2x) \, dx = \frac{1}{2} \log |\sec(2x)| + c. \)  

The graph of \( f(x) = \frac{1}{2} \arctan(x) \) is shown below.

![Graph of \( f(x) = \frac{1}{2} \arctan(x) \)]

b. i. Write down the equations of the asymptotes.  

ii. On the axes above, sketch the graph of \( f^{-1} \), labelling any asymptotes with their equations.
c. Find $f(\sqrt{3})$.  

1 mark

__________________________________________________________________________

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d. Find the area enclosed by the graph of $f$, the $x$-axis and the line $x = \sqrt{3}$.  

2 marks

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Question 9 (6 marks)

Consider the curve represented by $x^2 - xy + \frac{3}{2}y^2 = 9$.

a. Find the gradient of the curve at any point $(x, y)$. 2 marks

b. Find the equation of the tangent to the curve at the point $(3, 0)$ and find the equation of the tangent to the curve at the point $(0, \sqrt{6})$.
Write each equation in the form $y = ax + b$. 2 marks

c. Find the acute angle between the tangent to the curve at the point $(3, 0)$ and the tangent to the curve at the point $(0, \sqrt{6})$.
Give your answer in the form $k\pi$, where $k$ is a real constant. 2 marks
SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Instructions

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.
Specialist Mathematics formulas

Mensuration

area of a trapezium: \( \frac{1}{2}(a + b)h \)

curved surface area of a cylinder: \( 2\pi rh \)

volume of a cylinder: \( \pi r^2h \)

volume of a cone: \( \frac{1}{3}\pi r^2h \)

volume of a pyramid: \( \frac{1}{3}Ah \)

volume of a sphere: \( \frac{4}{3}\pi r^3 \)

area of a triangle: \( \frac{1}{2}bc \sin A \)

sine rule: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)

cosine rule: \( c^2 = a^2 + b^2 - 2ab \cos C \)

Coordinate geometry

ellipse: \( \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \)

hyperbola: \( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \)

Circular (trigonometric) functions

\( \cos^2(x) + \sin^2(x) = 1 \)

\( 1 + \tan^2(x) = \sec^2(x) \)

\( \cot^2(x) + 1 = \csc^2(x) \)

\( \sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y) \)

\( \sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y) \)

\( \cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y) \)

\( \cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y) \)

\( \tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)} \)

\( \tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)} \)

\[ \cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \]

\[ \sin(2x) = 2 \sin(x) \cos(x) \]

\[ \tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)} \]

<table>
<thead>
<tr>
<th>function</th>
<th>( \sin^{-1} )</th>
<th>( \cos^{-1} )</th>
<th>( \tan^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
<td>([-1, 1])</td>
<td>([-1, 1])</td>
<td>( R )</td>
</tr>
<tr>
<td>range</td>
<td>([\frac{-\pi}{2}, \frac{\pi}{2}])</td>
<td>([0, \pi])</td>
<td>( (-\frac{\pi}{2}, \frac{\pi}{2}) )</td>
</tr>
</tbody>
</table>
Algebra (complex numbers)

\[ z = x + yi = r(\cos \theta + i \sin \theta) = r \ cis \theta \]

\[ |z| = \sqrt{x^2 + y^2} = r \]

\[ z \overline{z} = r_1 r_2 \ cis(\theta_1 + \theta_2) \]

\[ z^n = r^n \ cis(n\theta) \] (de Moivre’s theorem)

Calculus

\[ \frac{d}{dx} (x^n) = nx^{n-1} \]

\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, \quad n \neq -1 \]

\[ \frac{d}{dx} (e^{ax}) = ae^{ax} \]

\[ \int e^{ax} \, dx = \frac{1}{a} e^{ax} + c \]

\[ \frac{d}{dx} (\log_e(x)) = \frac{1}{x} \]

\[ \int \frac{1}{x} \, dx = \log_e |x| + c \]

\[ \frac{d}{dx} (\sin(ax)) = a \cos(ax) \]

\[ \int \sin(ax) \, dx = \frac{1}{a} \cos(ax) + c \]

\[ \frac{d}{dx} (\cos(ax)) = -a \sin(ax) \]

\[ \int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + c \]

\[ \frac{d}{dx} (\tan(ax)) = a \sec^2(ax) \]

\[ \int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax) + c \]

\[ \frac{d}{dx} (\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \]

\[ \int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1} \left( \frac{x}{a} \right) + c, \quad a > 0 \]

\[ \frac{d}{dx} (\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}} \]

\[ \int \frac{-1}{\sqrt{a^2-x^2}} \, dx = \cos^{-1} \left( \frac{x}{a} \right) + c, \quad a > 0 \]

\[ \frac{d}{dx} (\tan^{-1}(x)) = \frac{1}{1+x^2} \]

\[ \int \frac{a}{a^2+x^2} \, dx = \tan^{-1} \left( \frac{x}{a} \right) + c \]

Product rule:

\[ \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

Quotient rule:

\[ \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

Chain rule:

\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \]

Euler’s method:

If \( \frac{dy}{dx} = f(x), \ x_0 = a \) and \( y_0 = b \), then \( x_{n+1} = x_n + h \) and \( y_{n+1} = y_n + hf(x_n) \)

Acceleration:

\[ a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dy}{dx} = \frac{d}{dx} \left( \frac{1}{2} v^2 \right) \]

Constant (uniform) acceleration:

\[ v = u + at \quad s = ut + \frac{1}{2} at^2 \quad v^2 = u^2 + 2as \quad s = \frac{1}{2} (u + v)t \]

TURN OVER
Vectors in two and three dimensions

\[ \mathbf{r} = xi + yj + zk \]

\[ |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = r \]

\[ \mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos \theta = x_1 x_2 + y_1 y_2 + z_1 z_2 \]

\[ \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \]

Mechanics

momentum: \[ \mathbf{p} = m\mathbf{v} \]

equation of motion: \[ \mathbf{R} = m\mathbf{a} \]

friction: \[ F \leq \mu N \]