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VCE Bulletin Supplement 2

VCE Mathematics application tasks

Introduction

The following suggested themes and related advice are provided to assist teachers in devising suitable application tasks for Further Mathematics Unit 3, Mathematical Methods Units 3 and Specialist Mathematics Unit 4 in 2003. Application tasks are particularly well suited to the use of investigative, modelling and problem solving approaches that involve the use of mathematics in real life contexts. The following suggested themes and contexts conform to the design parameters for an application task described in the *Mathematics Assessment Guide VCE 2003*. Teachers may use the starting points outlined below, or devise their own application tasks. All outcomes are to be covered by the application task, with an emphasis on Outcomes 2 and 3. The three components of the application task for each course are as follows:

Further Mathematics – a data analysis application task

1. Displaying and organising univariate and bivariate data.
2. Consideration of general features of the data.
3. Undertaking analysis of the data such as regression analysis, the use of transformations to linearity, de-seasonalisation or analysis of time series.

Mathematical Methods – a function and calculus application task, and Specialist Mathematics – a problem solving or modelling application task

1. Introduction of a context through specific cases or examples.
2. Consideration of general features of this context.
3. Variation, or further specification, of assumptions or conditions involved in the context to focus on a particular feature related to the context.

Themes and contexts

Further Mathematics – Theme: Issues and opinions on globalisation

Increasingly, different countries and peoples of the world are coming to view themselves as part of a global community. However, within the global community, there can often be quite distinct differences between societies. The consideration and analysis of various data sets can reveal relationships between factors in a society that can then inform views and decision-making processes. They can also be used to reveal trends in terms of both social and economic factors.

A suitable data set that can be used in conjunction with the following possible starting points for an application task that investigates aspects of the suggested theme is the World Data, 1997 from UNESCO.

This data set can be accessed from the VCAA website at: www.vcaa.vic.edu.au/VCE/STUDIES/maths/maths.htm and provides information on the following variables:

Variable name	Description
<i>COUNTRY</i>	The name of 100 countries from around the world
<i>LITERACY</i>	The literacy rate (%) for the country
<i>LITMALE</i>	The literacy rate (%) for males for the country
<i>LITFEMA</i>	The literacy rate (%) for females for the country
<i>RADIO</i>	The number of radios per 1000 people in the population
<i>TV</i>	The number of TVs per 1000 people in the population
<i>GDP</i>	The gross domestic product per head of population
<i>LOGGDP</i>	The log (base 10) of the GDP per head of population
<i>LITGROUP</i>	The literacy rate of the country recorded as Low (1), Medium (2) or High (3)
<i>GDPGROUP</i>	The GDP of the country recorded as Low (1), Medium (2) or High (3)
<i>OECD</i>	Whether the country is a member of the OECD recorded as Yes (1) or No (2)

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of:

- the standard statistical terms and techniques used to display, summarise and describe univariate data for both categorical and numerical data;
- the concept of sample and population and the use of random numbers as a means of selecting a simple random sample of data from a population;
- the standard terms and techniques used to display and describe associations in bivariate data for both categorical and numerical data;
- the technique of regression as a means of modelling the relationship between two numerical variables with a straight line;
- the role of residual analysis and the coefficient of determination in making decisions about the appropriateness of a particular regression model;
- the concept of data linearisation through transformation;
- the terms used to describe standard patterns in time series in qualitative terms, the role of smoothing in helping to identify these patterns, and some simple techniques for quantifying these patterns;
- the assumptions and/or limitations that underlie the applications of statistical techniques.

All aspects of Outcome 2 and Outcome 3 are relevant.

Starting point 1: Investigating literacy rate and wealth

This starting point focuses on the investigation of whether a relationship exists between literacy rate and wealth. The variables *LITERACY*, *GDP*, *LITGROUP* and *GDPGROUP* should be used.

Component 1: Selection of a random sample of the data, construction of a scatterplot showing literacy rates and GDP and construction of a cross-tabulation, appropriately percentaged, of the classification variables (*LITGROUP* and *GDPGROUP*).

Component 2: Summary and description of literacy rates (univariate analysis, shape of distribution, centre and spread of the distribution).

Component 3: Transformation of axes scales to linearise the relationship between independent and dependent variables (log GDP figures are given for convenience), calculation of correlation coefficient, coefficient of determination, and regression equation for linear relationship and interpretation of results with respect to the selected variables.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work are:

- use of a range of standard statistical techniques and terms to display, summarise, describe and interpret patterns in data, and outline the assumptions and/or limitations relating to the application of these skills;
- selection of a simple random sample from a given population using a table of random numbers or an alternative random number generator;
- use of the technique of linear regression to model a relationship between two numerical variables;
- interpretation of the parameters in a regression equation in relation to the situation being modelled;
- use of data transformations, where appropriate, to linearise a set of bivariate data as a means of improving the fit of a regression model.

The following outlines some other possible starting points for investigation that could be developed using the 1997 UNESCO World Data. For each starting point, several aspects for investigation that could be incorporated into the three components of an application task are given.

Starting point 2: Literacy rate and gender

- comparison of literacy rates for males and females for a selection of countries (comparison of shape, centre and spread);
- construction of a scatterplot to investigate the relationship between literacy rates for males and GDP, use of a suitable transformation to linearise the relationship (log GDP figures are given for convenience);
- construction of a scatterplot to investigate the relationship between literacy rates for females and GDP, use of a suitable transformation to linearise the relationship (log GDP figures are given for convenience);
- calculation of correlation coefficient, coefficient of determination, and regression equation for each of the linear relationships (males and females). Comparison and interpretation of data and statistics related to the variables.

Starting point 3: Comparison of OECD and non OECD countries

- use of the classification variable (GDPGROUP) to construct a cross-tabulation, appropriately percentaged, and to investigate the relationship between wealth and OECD membership;
- comparison of radio ownership for OECD and non OECD countries (comparison of shape, centre and spread);
- comparison of TV ownership for OECD and non OECD countries (comparison of shape, centre and spread);
- use of the classification variable (LITGROUP) to construct a cross-tabulation, appropriately percentaged, and to investigate the relationship between literacy and OECD membership;
- comparison and interpretation of literacy rates for OECD and non OECD countries (comparison of shape, centre and spread).

Starting point 4: Wealth

- construction of a scatterplot, to investigate the relationship between radio ownership and GDP;
- construction of a scatterplot to investigate the relationship between TV ownership and GDP;
- calculation of correlation coefficient, coefficient of determination, and regression equation for each of the linear relationships (radio vs GDP and TV vs GDP). Comparison and interpretation of data and statistics related to the variables.

Teachers are encouraged to research and use other data sets that may be relevant to the suggested theme, and develop investigations based around the suggested theme using their own starting points. In particular, teachers seeking to develop an investigation based around time series analysis will find suitable **time series data sets** at the World Bank website: <http://devdata.worldbank.org/hnpstats/query/countries.htm>.

Mathematical Methods – Theme: Taking it to the limit

In mathematical and other application situations, we often come across problems for which there is no readily accessible analytical method of finding a solution, and instead an approach involving approximations is required. This may involve solving an equation in a theoretical context, a modelling problem involving discrete data for which we want to estimate the rate of change, a practical problem where we have an area that we wish to estimate, or a shape the boundary

of which we wish to approximate by a set of joined linear segments.

For this theme a particular context involving functions, calculus and approximations would be developed. The context may require an equation to be solved, where the relationship between an approximate solution and the tangent to the curve at this approximate solution is investigated and used to derive a better approximation. The context may involve approximating a function by linear segments which are tangential to a curve at some point. Alternatively the context may involve approximating an area by smaller regular areas, and then reducing some dimension of the smaller areas to find a better approximation. It may involve estimating the rate of change of data given in a discrete form, and using the rate of change to estimate future values, or modelling the data with a suitable function and comparing the estimates of the rate of change with the rates of change given by the model.

Assumptions and key parameters used to set up an appropriate model or process, including domain and range restrictions, the relationship between functions and their rate of change or cumulative value, mathematical analysis and the interpretation of results and discussion of limitations of the model or process used are key elements of these starting points. In each case calculus would be involved in the determination of the rates of change or cumulative value of functions. Analytical, graphical and numerical approaches, including the use of technology, will all be relevant to each starting point.

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of:

- the key features and properties of a given graph of a function or relation;
- the relation between numerical, graphical and symbolic forms of information about functions and equations and the corresponding features of those functions or equations;
- the appropriate selection of a technology application in a variety of mathematical contexts;
- analytical or numerical approaches to solving to solving equations and the nature of corresponding solutions;
- the tangent to a curve at a given point;
- the concept of approximation to the area under a curve using rectangles.

All aspects of Outcome 2 and Outcome 3 are relevant.

Starting point 1: Finding the roots of an equation

Many applications of mathematics lead to equations for which there is no easy way of finding the roots. For example, we may wish to find two numbers whose sum is 6 and whose product is 2. The exact answers of $3 \pm \sqrt{7}$ are not in a particularly useful form for practical purposes. Alternatively, we may wish to find the coordinates of the point P on the parabola $y = x^2$ which is closest to the point (2, 1). Again, the exact value of the real solution to this problem is not in a particularly useful form for practical purposes. In other circumstances, if the purchase price of a car is quoted as \$15000 or \$350 per month for five years, we may wish to determine the monthly interest rate.

For a general linear equation of the form $ax + b = 0$, or quadratic equation of the form $ax^2 + bx + c = 0$, where a , b and c are real numbers with $a \neq 0$, there are well-known formulas for the roots of these equations. There are also

formulas for the roots of general polynomial equations of degree 3 and degree 4, but these are extremely complicated and do not readily provide numerical values that may be required in practical application situations. If p is a general polynomial function of degree greater than or equal to 5, there is no such formula. There is also no formula for the exact roots of equations such as $\sin(x) = x - 1$.

In this starting point, the use of tangents to curves to find roots will be considered. The process used is an iterative numerical method, that requires an initial estimate (this can be obtained from a graph of the function, or functions, involved in setting up the equation), and then uses this value to find a better estimate. In general, the process works as follows:

- Step 1: for each root of the equation, start with an initial estimate of the root.
- Step 2: for the corresponding point on the curve, find the equation of the tangent line to the curve at this point, and find where this tangent cuts the x -axis.
- Step 3: use this as the new estimate of the root and repeat the procedure.

An alternative method is to use secants to a curve, based on two initial estimates, x_{first} and x_{second} of the root to the equation:

- Step 1: begin with two estimates of the root, x_{first} and x_{second} , these give two corresponding points on the graph of the function which are used to determine the equation of the line (the secant line) through these two points.
- Step 2: find the point where this line cuts the x -axis, this gives a new estimate of the root, call it x_{new} .
- Step 3: repeat the procedure, with x_{second} becoming x_{first} and x_{new} becoming x_{second} so that successive values of x_{new} approach the root.

Component 1: Selection of context and determination of an appropriate equation whose solution is to be found. This may be of the form $f(x) = k$, for some real number k , or of the form $f(x) = g(x)$ for suitable functions f and g . Sketch of the graph of the function, and estimation of the required solutions from the graph. Use of an appropriate simple approach, for example, zooming in on the graph, or tables of values, to estimate the solutions to a specified degree of accuracy. Use of a built in numerical method using technology, for example, 'zero', 'intersect', or 'solve' on graphics calculator, to determine the values of the roots to a specified degree of accuracy.

Component 2: Determination of the roots of an equation using the tangent method. Consideration of when to halt the process in terms of the number of iterations required to obtain an approximation to a specified accuracy.

Component 3: Consideration of what problems may occur when using this method. For example, consideration of the case where the slope of the tangent at the initial estimate is close to zero and/or how close an initial estimate of the root/s needs to be so that the method gives an approximation to the anticipated

root. Comparison of the tangent method to the similar method based on secants. Comparison of the two methods of finding the roots in terms of the number of iterations required to achieve the desired accuracy, effect of initial guesses, and so on, for a variety of equations.

Alternatively, computers take significantly longer to carry out divisions than multiplications and additions. Application

of the tangent method to, $\frac{1}{x} - a = 0$ and derivation of the algorithm $x_{n+1} = 2x_n - ax_n^2$ for successive approximations in determination of the reciprocal of a given number, for example, use of this algorithm for approximation of the reciprocal of an integer, such as 23, correct to 6 decimal places; or application of the tangent method to the equation $x^2 - a = 0$, and derivation

of the algorithm $x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$, as used by the ancient

Babylonians for approximation of \sqrt{a} , could be investigated.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions;
- secants and tangents to curves;
- use of calculus to determine equations of tangents;
- informal consideration of rate of convergence of an approximation to a known root;
- features which may impede successful termination of an algorithm.

Starting point 2: First order linear approximations

The relationship $f(x+h) \approx hf'(x) + f(x)$, for suitably small values of h , can be used to determine approximate values for the function f near x in terms of a small changes h in the value of x , and the values of the function and its derivative at x . This process is illustrated and applied in the investigation of straight edged boards surrounding the edges of a swimming pool. This involves both numerical and analytical techniques and an informal consideration of error. The relationship between a curve and tangents to the curve at given points is necessarily a part of this investigation. This relationship can be explored through the use of a family of linear functions, g_a defined for different intervals centred around a given point.

Component 1: Description of an edge, or boundary, of a swimming pool by a curve, such as a parabola with equation $f(x) = 0.01x^2$ with $-20 \leq x \leq 20$, where the units of measurement are in metres. Consideration of the case where a builder wishes to use rectangular boards to lie along the edge of the pool so that the mid point of one side of a board is tangential to the edge of the pool at a given edge and where the board is not 'too far' from the boundary. Use of the first order approximation formula:

$$f(x_0 + h) \approx hf'(x_0) + f(x_0)$$

for a point $(x_0, f(x_0))$ to develop a new

function $g(x)$, where $x = x_0 + h$ and $g(x) = (x - x_0)f'(x_0) + f(x_0)$.

Consideration of different board lengths such that one edge of the board is tangential at a particular point, such as when $x = 10$. For example, board edges defined by the family of functions:

$$g_a : [10 - a, 10 + a] \rightarrow R,$$

$$g_a(x) = (x - 10)f'(10) + f(10)$$

including analysis of the error measure $f(x) - g_a(x)$ for $x \in [10 - a, 10 + a]$. The length of a given board can be calculated.

Component 2: Consideration of different ways of laying boards along the edge of the pool. Investigation of boards placed at several other points in the interval $[0, 20]$, including the use of symmetry to obtain results about the interval $[-20, 0]$. Consideration of ‘fitting’ boards along the edge of the entire pool, including the similarity and difference for the errors involved for different selected points.

Component 3: Investigation of other curves defining the edge, or boundary, of a pool, for example a simple cubic polynomial function with the rule $f(x) = 0.0005x^3$ on the interval $[0, 20]$. Comparison between different curves of errors obtained for a range of different points.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions;
- use of calculus to determine a linear approximation to a function near a point;
- approximation of a shape by a series of linear segments;
- consideration of error in approximation of the boundary of a shape by linear segments.

Starting point 3: Approximating areas

Where the areas of various shapes or regions are required to be determined, this can be done using a variety of techniques. In some cases there will be known formulas for standard shapes, for other cases integration can be used where the shape whose area is to be determined can be readily described by curves that are part of the graph of known functions and for which anti-derivative functions can be found. In yet other cases numerical approximations based on sums of area of regular shapes that approximate the area of the required shape or region can be used. In this starting point determining areas by the use of sums of areas of left and right rectangles for a given function over an interval is investigated. This technique can be used to find the areas of plane shapes generated in practical situations, such as architectural features, swimming pools, sails, areas of land, furniture or surfaces of sculptures, to a required accuracy.

For a given function f , the left and right rectangle sums are determined as follows. Consider the function f with rule $f(x)$ over the interval $[a, b]$. Let the interval be partitioned into

n sub-intervals of equal length $\frac{b-a}{n}$ by the ordered sequence of points $a = x_0, x_1, x_2, \dots, x_n = b$.

The left and right rectangle sums for n sub-intervals, denoted by L_n and R_n respectively, are given by:

$$L_n = \frac{(b-a)}{n} \sum_{i=0}^{n-1} f(x_i) \text{ and } R_n = \frac{(b-a)}{n} \sum_{i=1}^n f(x_i)$$

Component 1: Consideration of a positive continuous function f with positive gradient over an interval $[a, b]$ and for which $\int_a^b f(x)dx$ can be determined exactly, such as $f(x) = x^2$ over the interval $[1, 2]$. Determination of approximations for the area of the region with boundaries $x = a, x = b$ the x axis and the curve $y = f(x)$ using four left and right rectangles. Evaluation of $R_4 - L_4$ and explanation of the relationship

$L_4 \leq \int_a^b f(x)dx \leq R_4$. Consideration of the corresponding difference and inequality for larger values of n .

Component 2: Demonstration that

$$R_4 - L_4 = \frac{b-a}{4} (f(x_4) - f(x_0))$$

where the partition of $[a, b]$ is given by $a = x_0, x_1, x_2, x_3, x_4 = b$. Consideration of a similar expression for $R_n - L_n$ where n rectangles are formed. Investigation of the error in using rectangles to approximate the value of an integral, including determination of the smallest value of n to produce a required accuracy, for example, where the error is at most 0.1 for the rule $f(x)$ chosen in Component 1. Determination of the smallest value of n for which the error is at most a given positive real value ϵ . Investigation of other functions for which an anti-derivative is known, in particular the case of a function for which the gradient is negative over an interval $[a, b]$.

Component 3: Approximation of the value of $\int_a^b g(x)dx$ for a function, g , where an antiderivative for $g(x)$ is not known by the student, using left and right rectangle sums, such as $\int_1^2 g(x)dx$ where $g(x) = (1+x^2)^{\frac{1}{3}}$ and the error is required to be less than 0.1. Application of such functions to finding areas of shapes or regions from practical situations such as a sunshade sail, a ‘kidney shaped’ swimming pool or an island. For example, consider an island described by the region contained between the curve with rule $y = g(x)$ and the curve with rule $y = 3 - g(x)$.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions;
- use of rectangles to approximate areas;
- consideration of error associated with approximation;
- choice of rectangles of appropriate width to estimate an area to a given accuracy.

Starting point 4: Numerical differentiation

Many situations provide discrete data for analysis, such as the annual population of Australia over several years, daily exchange rates, the distance of an object from a fixed reference point, as measured by a digital device and growth of money in a savings account. From this data we may wish to determine: the population growth rate, the rate of change in exchange rates, the speed of the object or the rate of growth of the money. Since the data is not continuous, we cannot determine an instantaneous rate of change, but we can estimate the rate of change at a point from the data. There are two common ways of determining the average rates of change for discrete data. The first is the forward difference approximation, and the second is the centred-difference approximation used by many calculators to provide numerical values for the derivative of a function at points in its domain.

Consider discrete data for the values of a function f at $x_0, x_1, x_2, \dots, x_n$, where the x_i are equally spaced across the domain interval from which the data is drawn, that is, there is an equal difference $h = x_{i+1} - x_i$ for $i = 1, 2, \dots, n$. The forward - difference and centred - difference approximations for $f'(x_i)$ are respectively given by:

$$\frac{f(x_{i+1}) - f(x_i)}{h} \text{ and } \frac{f(x_{i+1}) - f(x_{i-1}))}{2h}$$

Reliable and accurate estimation of population growth is very important for planning purposes. This starting point investigates change in a population over a period of time.

Component 1: Graphical representation of population data, for example in Australia, over a period of time, such as the last ten years, and determination of the average rate of change with respect to successive years of this data using the forward-difference and the centred-difference methods. Comparison of values for the average rate of change obtained.

Component 2: Determination of the average rate of change using each of the above methods for a variety of functions (including those which might be used to model the population data) over particular intervals for a given step size. Comparison of these with the instantaneous rate of change to identify which method is more accurate.

Component 3: Investigation of different step sizes, including as step sizes decrease, and investigation of problems that may occur, such as with functions used to model situations that may not be continuous or differentiable at some points in the domain. Determinations of the relative rate of change (the rate of change divided by the population at that time) for population data,

and use of this to predict the population over the next five years. Selection of an appropriate continuous function model for trend in population growth, and determination of the exact rate of change at the end of each year, for this model. Comparison of these with numerical estimates of rates of change. Use of these to predict the Australian population over the next five years and consideration of differences between models.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- appropriate representation of discrete data;
- use of numerical methods to estimate rates of change;
- graphical representation and analysis of functions;
- use of calculus to determine rates of change;
- identification of features of functions which would make the forward-difference or centred-difference method more accurate in estimating the rate of change.

Specialist Mathematics – Theme: From this into that

An important mathematical activity is the transformation of mathematical objects from one form or representation to an equivalent or related form or representation. For example, substitution and other techniques may be applied to transform a complicated integral into a simpler one of a recognisable standard form; complex numbers may be represented in polar form for ease of certain computations; graphs of three-dimensional parametric curves may be projected onto two-dimensional planes and the rules of functions may be expressed in alternative forms for ease of graphical analysis. It is important to consider what assumptions or constraints might apply for the transformation process to be valid, as well as any limitations on results. For this theme, a particular context where students investigate and analyse a situation involving functions, graphs, transformations and calculus, as applicable, would be developed.

The suggested theme for the application task is particularly appropriate for work on functions and relations and their graphs, algebra and calculus in both practical and theoretical situations that involve integration. In particular this will involve the setting up, evaluation, and manipulation of parametric forms, combinations of functions, derivatives, anti-derivatives and integrals. Technology will be particularly useful to support related graphical analysis.

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of:

- specified functions and relations, the form of their sketch graphs and their key features, including asymptotic behaviour;
- techniques for finding derivatives and anti-derivatives of functions, the relationship between the graph of a function and the graph of its anti-derivative functions and definite integrals;
- analytical, graphical, and numerical techniques for setting up and solving equations involving specified functions and relations.

All aspects of Outcome 2 and Outcome 3 are relevant.

Students will need to draw two- and three- dimensional graphs and compute repeated derivatives and anti-derivatives and combine expressions involving circular functions, as applicable. The use of suitable technology such as graphics calculators, spreadsheets or computer algebra systems will be particularly appropriate for carrying out these processes.

Starting point 1: The long and winding road

A popular infant’s toy used to develop motor skills consists of wooden balls which slide along stiff wire which has been bent into simple yet interesting shapes. These shapes include curves such as spirals, loops and straight lines. Consider a section of the wire that has been bent into a coil or spiral. The spiral has the overall shape of a right, truncated cone. Assume, for example, that the simple descending spiral makes six complete circuits from its uppermost point to its lowermost point. Let the radius at the uppermost point be r_1 and the radius at the lowermost point be r_2 . The height difference between the two points is h (see Figure 1).

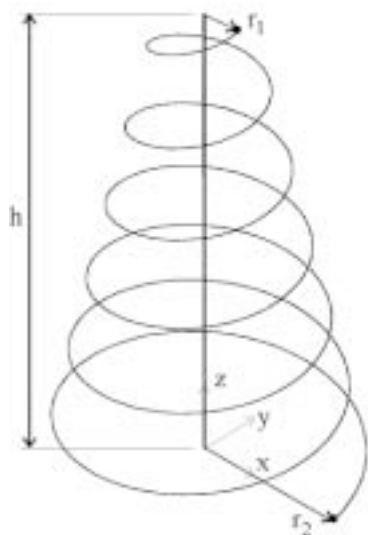


Figure 1: The spiral of six complete circuits

Component 1: Graphical representation the projection of the spiral on the horizontal x - y plane. Specification of a set of parametric equations that describes this two-dimensional projection of the spiral. Specification of a set of parametric equations that describes the three-dimensional path of the coil using Cartesian co-ordinates, for a given set of values for r_1 , r_2 and h .

Component 2: Consideration of the behaviour of the system for varying heights and numbers of circuits, for a given set of values of r_1 and r_2 , in particular the relationship between r_1 , r_2 , h , the number of circuits and the steepness of the path.

Component 3: For this component of the application task, a ball is released from the starting point at the top of the spiral section. Specification of a set of parametric equations

that describe the distance of the centre of the ball from the origin, perhaps as a function of θ , the angle of rotation, and graphical representation of this distance as a function of θ . Specification of a set of parametric equations that describe the distance of the centre of the ball from its starting point at the top of the spiral and graphical representation of this distance as a function of θ . Interpretation of these graphs and determination of the point on the path at which the ball is its maximum distance from the starting point.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- determining the specification of parametric equations to describe curves, including domain considerations;
- two- and three- dimensional graphs of parametrically specified relations;
- modelling distance as a function of angle of rotation and evaluation of maximum distance from a fixed point of reference.

Starting point 2: Recursion – we should do this again

In mathematics, recursion is a process that expresses a given stage of a procedure or construction in terms of earlier stages of the same procedure or construction. There are many examples of recursive processes in mathematics. For example, consider a function with rule of the form $f(x)e^x$, the derivative of this function is given by:

$$\frac{d(f(x)e^x)}{dx} = f'(x)e^x + f(x)e^x$$

$$\text{so } f(x)e^x = \frac{d(f(x)e^x)}{dx} - f'(x)e^x$$

$$\text{and hence } \int f(x)e^x dx = f(x)e^x - \int f'(x)e^x dx$$

If the last part of the previous expression, $f'(x)e^x$, is considered, and the same process applied with f' in place of f , the following new expression can be obtained:

$$\int f(x)e^x dx = f(x)e^x - f'(x)e^x + \int f''(x)e^x dx$$

This recursive process can be continued, in particular, if $f(x)$ is a polynomial function each application of the process would reduce the degree of the polynomial by one. A recursive formula involving integrals where each stage of the process reduces the complexity of the original integral is sometimes referred to as a *reduction formula*.

For this starting point the use of recursion and reduction formulas involving the evaluation of integrals and derivatives will be explored. These may arise in either in theoretical or practical contexts.

Component 1: Consideration of a function of the form $f(x) = x^n e^x$ for $n \geq 4$. Determination of the derivative and anti-derivative of f . Exploration of the relationship between various functions and their anti-derivatives, for example, functions with rules of the form $f(x) = x^n \sin(x)$ or $f(x) = \cos^n x$.

Component 2: Consideration of functions of the form described in Component 1 and expression of their anti-derivatives in general terms for integer values of n , for example by using descriptions such as: 'given $I_n = \int x^n e^x dx$ then $I_n = x^n e^x - nI_{n-1}$ ' (this should involve consideration of the general form for various functions of this type).

Component 3: Consideration of functions whose anti-derivatives can be described by reduction formulas. Identification and description of the nature of these functions. Alternatively, a similar approach could be applied to repeated differentiation of functions, for example, investigation of the relationship the derivatives of $\tan(x)$ and $\sec(x)$ and $\frac{d^n(\tan(x))}{dx^n}$, the n th derivative of $\tan(x)$ with respect to the variable x .

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- repeated differentiation and integration;
- recognition of recursive patterns;
- evaluation of derivatives and anti-derivatives using a variety of approaches including the possible use of technology.

Starting point 3: Curves into Squares and Triangles

In 1807, the mathematician Joseph Fourier announced that the rule for an arbitrary function defined over the interval $(-\pi, \pi)$ could, when certain conditions are satisfied, be

represented in the form: $\sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$. Such

expressions are called *Fourier series*. The coefficients in the expansion of *Fourier series* are given by the relations:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Using graphs for functions with rules represented in the form $\sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$, it is possible to represent various wave forms such as square waves and triangular waves, a somewhat surprising result given the curved nature of graphs

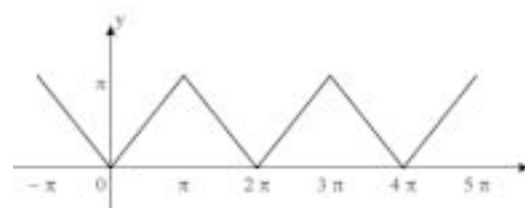
of the sine and cosine functions. For example, square waves may be represented by the *Fourier series*:

$$S(x) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=1,3,5,\dots} \left(\frac{\sin(nx)}{n^2} \right)$$



and triangular waves may be represented by the *Fourier series*:

$$T(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots} \left(\frac{\cos(nx)}{n^2} \right)$$



In this starting point waveforms based on square waves, triangular waves and combinations of these waveforms are investigated.

Component 1: Consideration of graphs of functions with rules of the form:

$$f_1(x) = a_1 \sin(x), f_2(x) = a_1 \sin(x) + a_2 \sin(3x),$$

$$f_3(x) = a_1 \sin(x) + a_2 \sin(3x) + a_3 \sin(5x) \text{ and so on for different combinations of values of the coefficients.}$$

Component 2: Consideration of the effect n has of the wave form for either $S(x)$ or $T(x)$, for example systematic analysis and comparison of the graphs of the functions such as:

$$S_1(x) = \sum_{n=1,3,5,\dots} (\sin(nx)), S_2(x) = \sum_{n=1,3,5,\dots} \left(\frac{\sin(nx)}{n^2} \right)$$

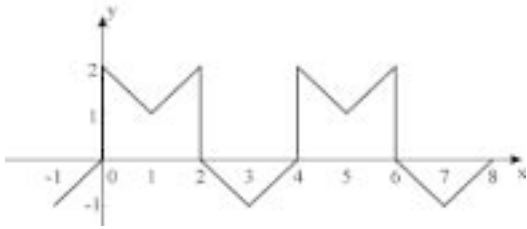
and

$$S_3(x) = -\frac{2}{\pi} \sum_{n=1,3,5,\dots} \left(\frac{\sin(nx)}{n^2} \right).$$

Investigation of changes so that the period and amplitude of the waves may be varied. Development of general forms of the rule so that waves of any period or amplitude can be graphed.

Component 3: Investigation of combinations of expressions for square and triangular waveforms to develop other Fourier series with graphs that represent wave forms such as waveforms A and B shown below, or other similar waveforms.

Waveform A



Waveform B



Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- specification of domains and ranges under the restrictions of the specified context;
- evaluation of definite integrals;
- analysis of the effects of parameters on the nature of the graph of a function;
- analysis of combinations of circular functions and the relationship of the rules of these combined functions to their graphs.

NOTES

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