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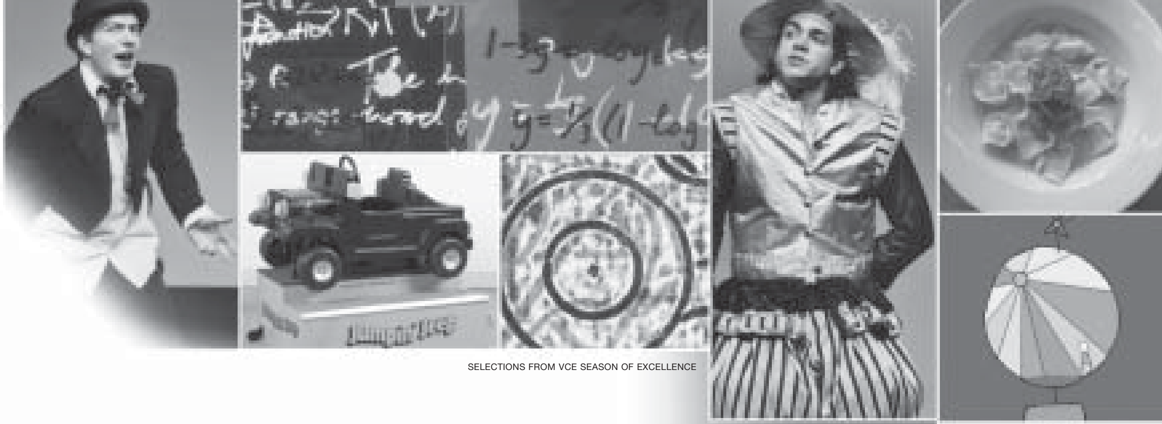

SELECTIONS FROM VCE SEASON OF EXCELLENCE

VCAA Bulletin Supplement 1

VCE Mathematics application tasks

Introduction

The following suggested themes and related advice are provided to assist teachers in devising suitable application tasks for Further Mathematics Unit 3, Mathematical Methods Unit 3 and Specialist Mathematics Unit 4 in 2004. Application tasks are particularly well suited to the use of investigative, modelling and problem-solving approaches that involve the use of mathematics in real life contexts. The following suggested themes and contexts conform to the design parameters for an application task described in the *VCE Mathematics Assessment Guide.* Teachers may use the starting points outlined below, or devise their own application tasks. All outcomes are to be covered by the application task, with an emphasis on Outcomes 2 and 3. The three components of the application task for each course are as follows:

Further Mathematics – a data analysis application task

1. Displaying and organising univariate and bivariate data.
2. Consideration of general features of the data.
3. Undertaking analysis of the data such as regression analysis, the use of transformations to linearity, de-seasonalisation or analysis of time series.

Mathematical Methods – a function and calculus application task
Specialist Mathematics – a problem-solving or modelling application task

1. Introduction of a context through specific cases or examples.
2. Consideration of general features of this context.
3. Variation, or further specification, of assumptions or conditions involved in the context to focus on a particular feature related to the context.

Themes and contexts

Further Mathematics – Theme: Exchange rates

Daily news services present a range of financial reports that are intended to be indicative of the health of the Australian economy. Reports are given on Stock Exchange index fluctuations, the price of gold and crude oil, and the value of the Australian dollar against a number of international currencies. Although the daily fluctuations in these measures may have little impact on the lives of most people, upward and downward trends can have a significant effect on how the economy's health is viewed. This is particularly true for the value of the Australian dollar.

Upward and downward trends in the value of the Australian dollar can be beneficial or detrimental to different sectors of the economy. When the dollar rises in value, imported goods become relatively cheaper to the benefit of consumers. Local manufacturers, however, are disadvantaged. When the dollar falls in value, imports become more expensive. However, with a falling dollar, Australian exports become more attractive to overseas purchasers, which is beneficial to exporters (for example, many of the primary industries).

Australians travelling overseas also need to be aware of currency fluctuations. Travellers wishing to purchase overseas currencies should do so when the Australian dollar is at a high relative value. Upon return to Australia, foreign currencies should be sold when the Australian dollar is at a low relative value.

Regardless of these mixed outcomes for different sections of the community, it is generally considered that the economy is performing well when the dollar is rising. As a country we feel good when it is rising. When it is falling or valued lowly we feel that we are underperforming economically.

Do the exchange rates vary enough to justify these reactions? Is the value of the Australian dollar showing any trends or patterns over a given period of time that can be useful predictors of future value in the short, medium or long term?

A suitable data set that can be used in conjunction with the following starting points can be accessed from the VCAA website at <www.vcaa.vic.edu.au/VCE/STUDIES/maths/maths.htm>. The data provides information on the value of the Australian dollar with respect to the following currencies:

Canadian dollars	Singapore dollars
Danish kroner	South African rand
Hong Kong dollars	Sri Lankan rupees
Indian rupees	Swedish kroner
Japanese yen	Swiss francs
Maltese pounds	Thai bahts
New Caledonian francs	US dollars
New Zealand dollars	UK pound
Norwegian kroner	

The data given is the means of the daily exchange rates over a period of a month. The set covers the time period from June 1983 to May 2003 inclusive.

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of:

- the standard statistical terms and techniques used to display, summarise and describe univariate data for numerical data
- the concept of sample and population and the use of random numbers as a means of selecting a simple random sample of data from a population
- the standard terms and techniques used to display and describe associations in bivariate data for numerical data
- the technique of regression as a means of modelling the relationship between two numerical variables with a straight line
- the role of residual analysis and the coefficient of determination in making decisions about the appropriateness of a particular regression model
- the concept of data linearisation through transformation
- the terms used to describe standard patterns in time series in qualitative terms, the role of smoothing in helping to identify these patterns, and some simple techniques for quantifying these patterns
- the assumptions and/or limitations that underlie the applications of statistical techniques.

All aspects of Outcome 2 and Outcome 3 are relevant.

Starting Point 1: Investigating exchange rates

Many individuals and sectors in the community are interested in both the daily values and trends in the value of the Australian dollar against overseas currencies. Small changes in the exchange rate can have a large impact on international transactions. Individuals planning overseas travel and companies involved in foreign trade are also keen to know of any trends in the value of the dollar. Accurate prediction of any trends can create a large financial benefit.

The aim of this starting point is to investigate the value of the Australian dollar against foreign currencies and the identification and analysis of any trends.

The value of the Australian dollar against the US dollar is investigated along with at least one other currency. Identification of time series patterns and trends are modelled by appropriate means which may include deseasonalisation, mean and median smoothing, and regression analysis.

Component 1: Selection of a random sample of data for US dollar exchange rate and other selected currencies. Sample selection made to provide a minimum of six years of continuous data for investigation. Construction of time-series plots and identification of any general trends or patterns.

Component 2: Description of the data in the plot. Calculation of summary statistics for central tendency and spread of data to provide an overview of the general nature of the data. Production of box plots, with consideration of outliers, for discussion of the spread of data. Consideration of the limitations of summary statistics in time series analysis.

Component 3: Investigation of data trends including consideration of possible seasonal, cyclic and trend patterns. Development of mathematical models describing the identified patterns, including trend lines with analysis on the fit of the regression models. Consideration of the value of different models as predictive instruments.

Key assessment features

The following key knowledge for Outcome 1 is particularly appropriate for this starting point:

- the standard statistical terms and techniques used to display, summarise and describe univariate data
- the concept of sample and population and the use of random numbers as a means of selecting a simple random sample of data from a population
- the standard terms and techniques used to display and describe associations in bivariate data for numerical data
- the technique of regression as a means of modelling the relationship between two numerical variables with a straight line
- the role of residual analysis and the coefficient of determination in making decisions about the appropriateness of a particular regression model
- the concept of data linearisation through transformation
- the terms used to describe standard patterns in time series in qualitative terms, the role of smoothing in helping to identify these patterns, and some simple techniques for quantifying these patterns
- the assumptions and/or limitations that underlie the applications of statistical techniques.

Important aspects of mathematics to be considered in assessment of student work are:

- use a range of standard statistical techniques and terms to display, summarise, describe and interpret patterns in data, and outline the assumptions and/or limitations relating to the application of these skills
- obtain a simple random sample from a given population using a table of random numbers or an alternative random number generator
- use the technique of linear regression to model a relationship between two numerical variables
- interpret the parameters in a regression equation in relation to the situation being modelled
- use one of the listed data transformations, where appropriate, to linearise a set of bivariate data as a means of improving the fit of a regression model
- display a time series (which may have been smoothed where necessary) graphically and use this to identify and describe any possible trend patterns
- use a range of simple techniques to describe features such as seasonality and trend in time series.

The following outlines some other possible starting points for investigation that could be developed based on the exchange rate data set or alternative sources of relevant data. For each starting point, several aspects for investigation that could be incorporated into the three components of an application task are given.

2. Investigating the exchange rate and another economic indicator

There is a range of economic indicators that are regularly used as guides to the economic health of the country. They include measures such as Gross Domestic Product (GDP), Consumer Price Index (CPI), Housing Finance, Personal Finance, and Average Weekly Earnings (AWE).

This starting point investigates any relationships between the Australian dollar exchange rate (in US$) and one other economic indicator. Key activities that can be included in this investigation are:

- Selection of appropriate data samples. A minimum of six years of data should be included.
- Description of the distributions in the samples (comparison of shape, centre and spread).
- Construction of a scatterplot to investigate the relationship between $US exchange rate and the selected indicator, and production of a linear model (if applicable).
- Use of appropriate transformations to enable identification of a suitable mathematical model. Calculation of correlation coefficient, coefficient of determination, and regression equation for the data. Interpretion of findings, including discussion of the model's limitations.

Many economic indicator data sets suitable for use are available from the Australian Bureau of Statistics <www.abs.gov.au/> for a small cost.

3. Investigating the value of the Australian dollar in other currencies

There are many factors, both real (in terms of actual economic production) and perceived (such as market confidence) that affect the value of a currency. However, different economic conditions in different countries should lead to differences in the exchange rate trends of their currencies against the Australian dollar. It is easy to envisage a situation where one country experiencing economic hardship has a currency with a falling trend (that is, the relative value of the Australian dollar is rising) and another country, with favourable economic conditions, has a currency rising in relative value (that is, the value of the Australian dollar is declining).

The focus of this starting point is an investigation of the relationship between the values of two currencies in terms of Australian dollars. Key activities that can be included in this investigation are:

- Selection of appropriate data samples. A minimum of six years of data should be included.
- Description of the distributions in the samples (comparison of shape, centre and spread).
- Construction of a scatterplot to investigate the relationship between the two currencies and production of a linear model (if applicable).
- Use of appropriate transformations to enable identification of a best mathematical model. Calculation of correlation coefficient, coefficient of determination, and regression equation for the data. Interpretion of findings, including discussion of the model's limitations.

4. Investigating the relationship between the exchange rate and foreign trade

Exchange rate fluctuations *should* have an impact on foreign trade. As the Australian dollar falls in value our exports become relatively cheaper to overseas purchasers. Imports however, become more expensive.

Similarly, a rising Australian dollar makes exports more expensive to overseas purchasers and imports into Australia cheaper.

The focus of this investigation is a comparison of the exchange rates (with Australia) for two different countries with their monthly balance of trade figures (with Australia). Key activities that can be included in this investigation are:

- Selection of appropriate data samples. A minimum of six years of data should be included.
- Description of the distributions in the samples (comparison of shape, centre and spread).
- Construction of a scatterplot to investigate the relationship between the exchange rates and balances of trade. Production of a linear model (if applicable).
- Use of appropriate transformations to enable identification of best mathematical models. Calculation of correlation coefficients, coefficients of determination, and regression equations for the data. Interpretation of findings, including discussion of each model's limitations.

Teachers are encouraged to find other data sets and develop their own investigation based around the theme. Rob Hyndman's Time Series Data Library <www-personal.buseco.monash.edu.au/~hyndman/TSDL/> and the Australian Statistical Internet Sites page of the National Library of Australia <www.nla.gov.au/oz/stats.html> are useful sources of data.

Mathematical Methods – Theme: A mean value

Average values are often used in mathematics and statistics and their applications. The most well known of these is the mean of a finite set of numbers, and the terms *mean* and *average* are generally used synonymously in everyday language. Thus we refer to such things as the average daily maximum temperature in a month, the average speed on a trip and the average number of children in a family.

It is a straightforward computation to calculate the average value of finitely many numbers using simple arithmetic operations. On the other hand, many variables are continuous, and a different approach will be needed to, for example, calculate the average temperature during a day where infinitely many temperature readings are, in principle, possible over the temperature range for that day. Similar considerations apply to finding an average value for the speed of a car over an interval of time during a trip, or an average value for the volume of inhaled air in the lungs in one respiratory cycle. The mean value theorem for integration provides a way to tackle this sort of problem. Using this theorem, the average value of a continuous function over a given interval is calculated by dividing the value of the definite integral of the function over that interval by the length of the interval. If a function is non-negative over the interval from $x = a$ to $x = b$, where a and b are real numbers with $a < b$, then the area between the graph of the function and the horizontal axis over this interval, is the same as that of a rectangle with base of length $b - a$ and height equal to the average value of the function.

The average value of the rate of change of a function over an interval can be readily determined. For example, the average rate of change of displacement, or average speed, for a car journey is calculated by dividing the distance travelled by the time taken for the journey. The mean value theorem for differentiation tells us that at some time during the trip, the instantaneous speed of the car will have been equal to the average speed. This explains the set up of speed cameras on major highways – although a car may not be 'speeding' instantaneously when it passes two cameras spaced a certain distance apart, if the distance between them is covered in a time that would make the average speed greater than the speed limit, then the driver could well be booked for speeding courtesy of the mean value theorem for differentiation!

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of

- the key features and properties of a given graph of a function or relation or families of types of functions or relations
- the concepts of domain, range and asymptotic behaviour of functions
- features which enable the recognition of general forms of possible models for data presented in graphical or tabular form
- analytical or numerical approaches to solving equations and the nature of the corresponding solutions (real, exact or approximate) and the effect of domain restrictions
- features which link the graph of a function to the graph of the corresponding gradient function or its numerical values, the tangent and normal to a curve at a given point, how the sign and magnitude of the derivative of a function can be used to describe key features of the function and its derivative function
- the concept of approximation to the area under the curve using rectangles, the ideas behind the fundamental theorem of calculus and the relationship between the definite integral and area.

All aspects of Outcome 2 and Outcome 3 are relevant.

Starting point 1: Investigation of the mean value theorem for differentiation

In many areas of mathematics and its application to both theoretical and practical situations, we are interested in determining mean values. The *mean value theorem for differentiation* states that if f is a continuous function over the interval from $x = a$ to $x = b$, where a and b are real numbers with $a < b$, and differentiable on the open interval from $x = ax = b$, then there is some $x = c$ in the open interval such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. That is, the average rate of change of f over the closed interval is equal to the instantaneous rate of change of f at some point c in the open interval. The aim of this starting point is to investigate the application of the mean value theorem for differentiation to several functions.

Component 1: Consideration of a quadratic function, for example with rule $f(x) = 2x^2 - 3x + 4$, its graph over a suitable closed interval $[a, b]$, and the gradient of the chord joining the interval endpoints with coordinates $(a, f(a))$ and $(b, f(b))$. Determination of the value $x = c$ in the open interval (a, b) such that $f'(c)$ is equal to the gradient of the chord. Location of the point $(c, f(c))$ on the graph, and sketch of the tangent to the graph at this point. Generalisation of this result for an arbitrary quadratic function.

Component 2: Consideration of other functions, such as the function $f: R \setminus \{0\} \to R, f(x) = \frac{k}{x}$, where k is a non-zero real constant, over various intervals, and generalisation of this result for any interval.

Consideration of $\left[0, \frac{\pi}{2}\right] \to R, g(x) = \sin(x)$ to find possible real numbers N and M such that $N \le \frac{f(x_i) - f(x_j)}{x_i - x_j} \le M$ for all x_i, x_j in the interval $\left[0, \frac{\pi}{2}\right]$ where $x_i > x_j$. Similar consideration of quadratic functions, including description of suitable intervals.

Component 3: Development of the result that if f is a differentiable function on the open interval (a, b) with $f'(x) = 0$ for all x in this interval, then f is a constant function over this interval. Extension of this result to show that if f and g are differentiable functions on the open interval (a, b) with $f'(x) = g'(x)$ then the functions have a constant difference over this interval.

Alternatively: Investigation of the relationship between functions and their derivatives, initially with, for example, a quadratic function and its derivative. Description of the graph of the function when its derivative is negative, zero and positive. A function f is said to *increasing* on an interval if $f(x_i) < f(x_j)$ whenever $x_i < x_j$ for all x_i and x_j in the interval. Similarly, a function f is said to be *decreasing* on an interval if $f(x_i) > f(x_j)$ whenever $x_i < x_j$ for all x_i and x_j in the interval. Development of the results that for a function f which is continuous on the closed interval $[a, b]$ and differentiable on open interval (a, b):

i. if $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.

ii. if $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

Demonstration of this result using various functions.

Or, alternatively: Development of the result that, for functions f and g which are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and with $g'(x) \ne 0$ for all x on the open interval, that there is at least one value $x = c$ on the open interval such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

This result is called the *extended mean value theorem* and is attributed to the French mathematician Augustin Cauchy (1789–1857). Hint: consider the application of the mean value theorem to the function $F(x) = (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x)$

Application of the extended mean value theorem to various functions of interest, for example, where $f(x) = \sin(x)$ and $g(x) = x$, and investigation of the situation where the value of b approaches the value of a.

Or, alternatively: Investigation of the turning ratio of a projectile, for example the launch of a rocket. To investigate the motion of the rocket the technicians consider a ratio called the turning ratio. For a given time interval $[t_0, t_1]$, the function A, which gives the angle between the flight path and some fixed axis at time t, and the function D, which gives the distance travelled at time t respectively, the turning ratio, T, is defined by:

$$T = \frac{A(t_1) - A(t_0)}{D(t_1) - D(t_0)}$$

Selection of suitable functions A and D and investigation of T, such as the determination of maximum and minimum values for T for a given time interval.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions that clearly indicate key features
- the use of calculus to determine the derivative of a given function
- solution of equations
- formulation of general mathematical results/conjectures;
- consideration of domains.

Starting point 2: Smooth sailing

Suppose that A and B are locations at opposite sides of a bay, such as Port Philip Bay or Sydney Harbour. It could be conjectured as to whether it is, or is not, possible to sail a boat from A to B without ever sailing in the exact (straight line) direction from A to B. That is, is it possible to sail from A to B without the instantaneous line of motion ever being parallel to the straight line segment line connecting A and B?

Similarly, if the locations A and B are 100 km apart by road around the bay, and are connected by a highway with a speed limit of 100 km/hr is it possible for a car to leave A at

1 pm and arrive at B at 1.50 pm without exceeding the speed limit? In a beach sport event, when a competitor runs 100 metres in 12 seconds, will they achieve a speed of 8.33 m/sec at any time during the race? If two competitors run in the same race, both starting and finishing at the same time, is their speed the same at any point during the race?

Each of these situations can be considered as applications of the *mean value theorem for differentiation*. This starting point considers simple functions used to model such situations and investigates the application of the mean value theorem for differentiation to these situations.

Component 1: Selection of a suitable function f, such as a quadratic function, which is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and can be used as a model for one of the above situations, or a similar scenario. Consideration of the graph of the model function, the line segment connecting the interval endpoints $(a, f(a))$ and $(b, f(b))$, and the average rate of change, f_{ave}, over the interval. Determination of the derivative function f' and the value/s for $x = c$ in the open interval (a, b) for which $f'(c) = f_{ave}$. Geometric interpretation of these considerations with respect to a conjecture related to the situation being modelled.

Component 2: Investigation of other models for these situations, including consideration of whether it is always possible to find a value $x = c$ in the open interval (a, b) for which, $f'(c) = f_{ave}$ for example, whether there is a possible model which is not continuous or differentiable. Generalisation of these results.

Alternatively, investigation of the conditions under which the above result is generally true. That is, where there exists a point c in (a, b) for which $f'(c) = f_{ave}$. Consideration of the assumption of differentiability and the assumption of continuity.

Component 3: Investigation of one of the other scenarios, where students create their own model.

Alternatively, investigation of the sailing boat context, with A lying due east of B, to extend the result of the mean value theorem for differentiation, by showing that if also $f(a) = f(b)$, then there is a point c in (a, b) for which $f'(c) = 0$. (This is called Rolle's theorem.)

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions that clearly indicate key features
- the use of calculus to determine the derivative of a given function
- solution of equations
- formulation of general mathematical results/conjectures
- consideration of domains.

Starting point 3: Investigation of the mean value theorem for integration

It is relatively straightforward to calculate the average value for a finite set of real numbers using arithmetic operations. A different approach is required if we wish to find the average value of a function over a given interval. This approach is based on the *mean value theorem for integration*:

If f is continuous function on the closed interval $[a, b]$ where a and b are real numbers with $a < b$, then there is some $x = c$ in the open interval (a, b) such that $\int_a^b f(x)dx = f(c)(b-a)$. Then $f(c)$ is called the average value of f over the closed interval $[a, b]$. In this starting point the relationship between this definition and the definition of the mean of a finite set of numbers is investigated by estimating the area under the curve by rectangles.

Component 1: Selection of a continuous function f, for example, a quadratic function, over a closed interval $[a, b]$ where $f(x) \geq 0$, and evaluation of the definite integral $\int_a^b f(x)dx$. Demonstration that this definite integral can be written in the form $A(b-a)$, where A is a real number, and hence identification of $x = c$ such that $f(c) = A$. Graphical interpretation showing the area under the curve and the rectangle with base of length $(b-a)$ and height A.

Component 2: Investigation of other functions, without restriction on the sign of f. Consideration of whether the result holds if the function is not differentiable or if it is not continuous on the interval, for example, in the case of hybrid functions.

Component 3: Investigation of the relationship between this definition of the average value of a function and the notion of the average of a finite set of numbers. Consideration of a function for which the mean value theorem for integration holds by division of a closed interval $[a, b]$ into n subintervals, selection of a value of the function at any point in each of the subintervals (for example, the left endpoints, the right endpoints or the midpoints) and computation of the average of this set of numbers. Comparison of this to the average value of the function determined by integration, in particular as the number of subintervals is increased. This process is based on re-arrangement of expressions of

$$\frac{\sum f(x_i)}{n} \text{ in the form } \frac{\sum f(x_i)\delta x}{(b-a)} \text{ where } \delta x = \frac{(b-a)}{n}$$

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions that clearly indicate key features
- the use of calculus to determine the antiderivative of a given function

- solution of equations
- formulation of general mathematical results/conjectures
- consideration of domains
- approximating the area under a curve.

Starting point 4: Averaging things out

The average value of a function of a real variable, over a given interval can, given certain conditions, be calculated using definite integrals by the mean value theorem. In particular, where the function being considered is used to model the behaviour of a continuous variable, the corresponding average value can be interpreted with respect to the context under investigation, involving variables such as temperature, voltage, speed and volume.

If f is continuous function on the closed interval $[a, b]$ where a and b are real numbers with $a < b$, then there is some $x = c$ in the open interval (a, b) such that

$\int_a^b f(x)dx = f(c)(b-a)$. Then $f(c)$ is called the average value of f over the closed interval $[a, b]$. In this starting point the relationship between this definition and its application in modelling contexts in investigated.

Component 1: Selection and consideration of a continuous function f on an interval $[a, b]$, which is a model for one of the above or similar scenarios with $f(x) \geq 0$ on $[a, b]$ and evaluation of $\int_a^b f(x)dx$. Demonstration that this can be written in the form $A(b-a)$, where A is a real number, and hence identification of $x = c$ such that $f(c) = A$. Graphical interpretation showing the area under the curve and the rectangle with base of length $(b-a)$ and height A. For example, the temperature, T, in degrees Celsius, at any time t (in hours after midnight) in a particular day could be modelled by the function:

$$T(t) = 16 + 2\sin\left(\frac{\pi}{12}t - \frac{\pi}{2}\right)$$

and the average temperature between 6 am and 6 pm would be considered.

Component 2: Investigation of other modelling functions, without restriction on the sign of f. Consideration of whether the result holds if the function is not differentiable or if it is not continuous on the interval, for example, models defined in terms of hybrid functions (such as the spread of disease and the impact of a new treatment at a certain point in time).

Component 3: Investigation of a function model for household electricity (alternating current with frequency 50 Hertz) where the voltage, V, at any time t in seconds is given by $V(t) = A\sin(100\pi t)$, where A is a real number. Demonstration that the average value of a function with rule $f(t) = A\sin(t)$ over a single cycle is zero. Definition of the root mean square (RMS) value of the function f above by

$$f_{RMS} = \sqrt{\frac{\int_0^{2\pi} f(t)^2\,dt}{2\pi}}$$

and determination of the RMS value of $f(t) = \sin(t)$ over one cycle, where the corresponding integration is computed by technology, or by using the relationship $\sin^2(\theta) = \frac{1}{2}\left(1-\cos(2\theta)\right)$ to convert the definite integral to a standard form. Evaluation of A given a RMS household voltage of 240 volts.

Alternatively, Investigation of the average speed of a car for various modelling functions for its displacement, where if $s(t)$ is the displacement of a car on a journey at time t, then average speed of the car over the time interval $[t_1, t_2]$ is

$$\frac{\delta s}{\delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Demonstration that the average speed of a car over a time interval $[t_1, t_2]$ calculated this way is the same as the average of the speed function over the trip, where the average value of the speed function over the trip is:

$$v_{average} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} s'(t)dt$$

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions that clearly indicate key features
- the use of calculus to determine an anti-derivative of a given function.
- solution of equations
- formulation of general mathematical results/conjectures
- consideration of domains.

Specialist Mathematics – Theme: Right and parallel

Families of curves can be investigated in a range of theoretical and practical application contexts including those that involve consideration of lines or curves that are *parallel* or *perpendicular* to the curves of a particular family at given points. In some cases families of curves considered may themselves be considered to be orthogonal (at right angles) to other families of curves. Such considerations will involve working with tangents and/or normals to curves that have been specified using cartesian equations, vector parametric equations or complex numbers. Related practical application contexts may involve the reflection of light from surfaces, the graphical solution of differential equations that model various real life situations, or radiation from heated bodies.

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of:

- specified functions and relations, the form of their sketch graphs and their key features, including asymptotic behaviour
- techniques for finding derivatives and anti-derivatives of functions, the relationship between the graph of a function and the graph of its anti-derivative functions and definite integrals
- analytical, graphical, and numerical techniques for setting up and solving equations involving specified functions and relations.

All aspects of Outcome 2 and Outcome 3 are relevant.

Students will need to draw two and three dimensional graphs, where families of curves are specified by cartesian equations, parametric equations, or functions of a complex variable, and find tangents and normals to a given family of curves. The use of suitable technology such as graphics calculators, spreadsheets or computer algebra systems will be particularly appropriate for carrying out these processes.

Starting Point 1: We should reflect on that

Reflective surfaces are used widely in everyday life. Some of the basic principles of reflective surfaces are understood intuitively, and can be simply stated, for example, the law of reflection that: the *normal* to a reflective surface bisects the angle formed by the incident ray and the reflected ray ($\alpha = \beta$) as shown in Figure 1:

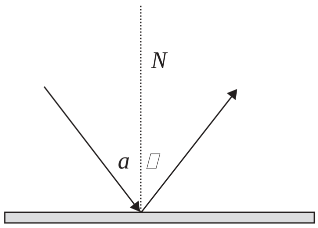

Figure 1: *the normal, N, to a reflective surface bisects the angle formed by the incident ray and the reflected ray ($\alpha = \beta$).*

For this starting point consideration will be given to the use of specific curved surfaces to control the reflection of light for a single source. It will be assumed that the light source is a point, and the system can be viewed as a two dimensional representation of a three dimensional application.

Component 1: Consideration of the curve of a reflecting surface defined by $\underset{\sim}{S} = at^2\underset{\sim}{i} + t\underset{\sim}{j}$, where a is an arbitrary real constant, where incident rays with equations $\underset{\sim}{r}(t) = t\underset{\sim}{i} + c\underset{\sim}{j}$ are reflected from the surface, for several values of the real constant c. Determination of the corresponding equations of the reflected rays and graphical representation of these with commentary on findings.

Component 2: Generalisation of results from Component 1. Application, using the cartesian form of the defining equation, to the design of parabolic mirrors that would be suitable for car headlights. Consideration of the location of the light source for both high and low beams.

Component 3: Investigation of elliptic reflecting surfaces (elliptic reflectors are used in medical equipment such as those that used ultra sonic wave to break down kidney stones. Waves that emanate from one focus of elliptic reflecting surface will pass through the other focus). This could be based on reflecting surfaces of the form $\underset{\sim}{S} = a\cos t\,\underset{\sim}{i} + b\sin t\,\underset{\sim}{j}$, given that the foci will occur at $(\pm\underset{\sim}{c}, 0)$ where c is a positive real number and $a^2 - b^2 = c^2$, leading to the design of an elliptic mirror that would deliver rays to a specific point. Interpretation in the modelling context for the design of such mirrors when used for medical purposes.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- determining the specification of parametric equations to describe curves, including domain considerations
- two and three dimensional graphs of parametrically specified relations.

Starting Point 2: Properties of functions of a complex variable

This starting point investigates the role of functions as maps of the complex plane. Attention should be paid to the relationship between the gradients of the tangents to curves in the complex plane defined as the image of subsets of the complex plane under a given function, at the point where the curves of the complex variables intersect.

Component 1: Familiarisation with functions of a complex variable. Consideration of the image of these functions when represented by a curve in the u-v plane describe by an equation in real variables u and v, where functions $f(z)$ are expressed in the form $u+iv$, with u and v real-valued functions of x and y. Consideration of a given function and graphing of functions developed from $\{z : z = x + ik, x \in R\}$ and $\{z : z = l + iy, y \in R\}$ for given values of l and k, where l and k are fixed real values.

Component 2: Development of equations expressing u in terms of v for a variety of values for l and k. Consideration of the gradient of the tangents of these curves at the point where curves intersect. Generalisation of results.

Component 3: Investigation of other functions of a complex variable such as the complex function with rule $f(z) = e^z$ or the complex function with rule $g(z) = \frac{1}{z}$. Consideration of the gradients of the tangents of the curves at the point where curves intersect. Conjecture about the relationship of the gradients of the tangents of curves at the point where curves intersect for functions of a complex variable of a particular type, for example complex functions with rules of the form $f(z) = az^2 + bz$ where a and b are real constants.

Starting Point 3: Hot bodies

Energy that is transferred between two bodies due to a difference in temperature is defined as heat. Heat may be transferred by conduction, convection or radiation. In designing any process or piece of equipment in which heat transfer may occur, engineers need to understand how heat is transferred, its rate and its direction.

For example, the behaviour of a solid body that is internally heat in some manner could be investigated, where it is assumed that every point on the surface is at the same temperature, that is, the surface of the body is *isothermal*. Suppose also that the body is completely submerged in a fluid having a temperature below the body's temperature. Because of the temperature difference between the body's surface and the fluid, heat will be transferred from the hotter body to the cooler fluid. So long as the surface of the body remains isothermal then the direction in which the heat leaves a point on the surface will be at right angles or *orthogonal* to the surface at that point. A family of curves which are orthogonal to a second family of curves are said to be *orthogonal trajectories* of one another.

Component 1: Consideration of two families of curves, where, for example, the first family of curves may be represented by the equation:

$xy = a$ Equation 1

and the second family of curves may be represented by the equation:

$x^2 - y^2 = b$ Equation 2

where a and b are non-zero real constants, and the first family of curves could represent a series of isothermal surfaces while the second family of curves could represent the flow of heat.

Graphical representation of both families of curves where $-5 \le x \le 5$ and $-5 \le y \le 5$ for several positive and negative real values of a and b. Demonstration that every curve of the family described by Equation 1 is orthogonal to every curve of the family described by Equation 2 for all non-zero real values of a and b.

Component 2: Identification of the lines described by Equation 1 when $a = 0$, and Equation 2 when $b = 0$. Consideration of a given point say, (x_1, y_1), on the x–y plane, which does not fall on either of those sets of lines and determination of the corresponding values for a and b defined by Equation 1 and Equation 2. Graphical analysis of the gradient function $\frac{dy}{dx}$, as a function of x, for both curves that pass through (x_1, y_1) and investigation of the behaviour of the system by consideration of a range of points.

Component 3: Consideration of the family of curves, where c is a real constant, represented by the equation:

$y^2 = -2(x + c)$ Equation 3

with orthogonal trajectories for this family of curves defined by the equation:

$y = ce^x$ Equation 4

and the first set of curves could similarly represent isothermal surfaces while the second set of curves could represent the corresponding directions in which the heat flows.

Graphical analysis of curves generated by both equations, for several values of c. Demonstration to confirm that the two families of curves are orthogonal.

Investigation of the family of curves represented by the equation:

$y = g x^2 + h$ Equation 5

where g and h are real constants, and determination of the orthogonal trajectories for this family of curves.

NOTES

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