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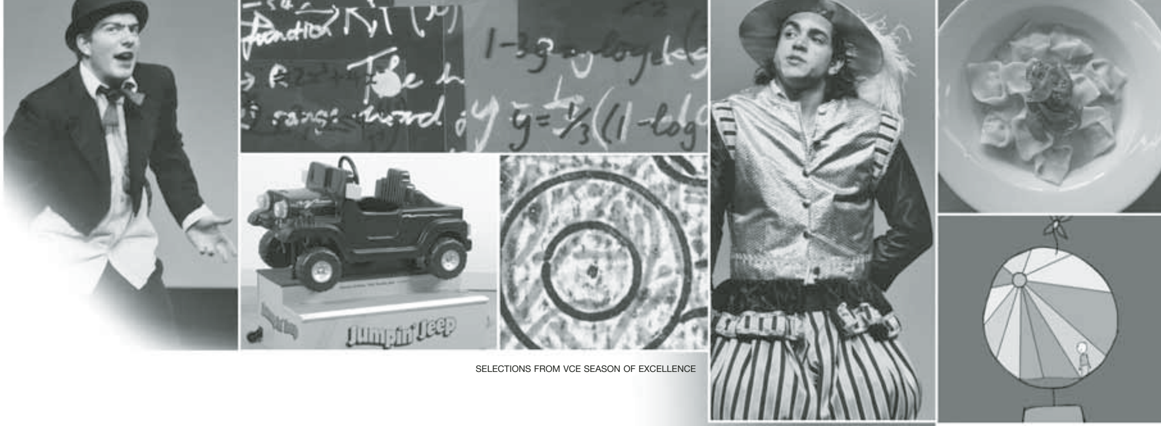

SELECTIONS FROM VCE SEASON OF EXCELLENCE

VCAA Bulletin Supplement 3

VCE Mathematics 2006–2009
Further Mathematics study advice – matrices

The revised VCE Mathematics study 2006–2009 includes a new module *Matrices* in the ‘Applications’ area of study for Further Mathematics Units 3 and 4.

The following provides advice on scope and depth of material for this new module, available to all schools offering Further Mathematics from 2006. While the *Matrices* topic in the ‘Arithmetic’ area of study for General Mathematics provides coverage of useful background material, it is intended that students will be able to undertake study of the *Matrices* module in Further Mathematics without having previously studied this topic.

The new module covers the matrix representation of discrete data in rectangular arrays, and the application of matrix arithmetic to the analysis of problems in practical situations. While students should be familiar with basic matrix computations by hand in simple cases, it is expected that technology will be used to carry out computations as applicable.

The *Advice for teachers* section for General Mathematics in both the current and revised study designs provide examples of suitable teaching and learning activities for the introduction of matrix representation and computation in a practical product inventory-sales context. The *Advice for teachers* section for Further Mathematics in the revised study 2006–2009 includes a detailed example covering matrix representation, the solution of simultaneous linear equations and transition matrices. The following advice should be read in conjunction with these materials.

Content

Module 6: *Matrices* from the ‘Applications’ area of study for Further Mathematics Units 3 and 4 includes the following content:

Matrix representation and its application including:

- matrix representation of data from a variety of situations in a rectangular array whose order is defined by the number of rows and columns in the array
- application of matrix arithmetic (sum, difference, scalar multiple, product) to solving practical problems, for example stock inventories, total value of sales, mark-ups and discounts
- application of simultaneous linear equations in practical situations, and their solution using the inverse matrix method.

Transition matrices including:

- application of simple transition matrices up to 4 × 4 to analyse practical situations such as consumer preferences in shopping, including an informal consideration of steady state (no noticeable change from one state to the next).

This content is to be developed in conjunction with the corresponding key knowledge and skills for Outcome 1:

Key knowledge

This knowledge includes:

- terms, symbols, notations and conventions for the representation of matrices (including vectors as row matrices or column matrices) and matrix operations
- matrix form of simultaneous linear equations and matrix inverse
- initial state, transition matrices and steady state.

Key skills

These skills include the ability to:

- represent data from situations using scalars, vectors (as column matrices or row matrices) and matrices
- use technology to calculate scalar multiples, sums and differences, products and inverses of matrices, including simple combinations of these operations as applicable
- solve systems of simultaneous linear equations using the matrix inverse method
- model situations using transitions matrices, and identify and/or calculate initial, transition and steady states in practical situations using technology.

In studying the *Matrices* module it is expected that students will develop a working knowledge of the following terms: matrix, vector (as a row matrix or column matrix), scalar, row, column, order, inverse, state, initial state, transition matrix and steady-state (state vectors should be represented as column vectors).

Students may choose to use either square or round brackets to represent matrices as two-dimensional arrays however *a single convention* should be used consistently within working for a given problem. Some technology applications may require the elements of a matrix to be entered as a *list* of *lists*, for example the 3 × 4 matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

may, for a particular technology, need to be entered in the form:

$\mathbf{A} = \{\{a_{11}, a_{12}, a_{13}, a_{14}\}, \{a_{21}, a_{22}, a_{23}, a_{24}\}, \{a_{31}, a_{32}, a_{33}, a_{34}\}\}$

or similar. Whatever form of representation is required for a given technology, students should use the conventional mathematical two dimensional representations of matrices in their written working. Matrix multiplication can be developed in practical situations using the notion of a sum of products of weighted values. Generally matrices of low order, with up to five rows or columns would be used in matrix computation, however, for modelling tasks and problems in coursework, and related assessment, higher order matrices could be considered, and students should use technology to carry out the relevant matrix computations as applicable.

Students should be able to perform *simple* computations on matrices of small order by hand, including matrix addition, subtraction, multiplication by a scalar and matrix multiplication. It is **not** intended that a general consideration of matrix algebra be incorporated into implementation of the module. However students should be aware of the basic properties of identity and inverse matrices and that matrix multiplication is not commutative, that is, in general, $\mathbf{AB} \neq \mathbf{BA}$.

Systems of simultaneous linear equations

Graphics calculators and CAS typically have built in functionality, or can load supplementary programs, for solving a $n \times n$ system of simultaneous linear equations for small values of n. More generally, if :

$\mathbf{AX} = \mathbf{B}$

is the matrix representation of a system of n simultaneous linear equations in n unknowns, then students should know that the solution of this system is given by:

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

They should be able to *formulate* systems of simultaneous linear equations amenable to this method of solution from data given in practical contexts, *apply* the method, and be able to *interpret* the solution with respect to these contexts, in cases where the matrix of coefficients **A** is non-singular (that is, $\mathbf{A}^{-1}$ exists). Students will **not** be required to consider systems of simultaneous linear equations which have infinitely many (parametric) solutions, or where the systems of simultaneous linear equations are inconsistent and thus do not have any solution. Although not required, teachers may wish to consider these cases informally.

While an explicit treatment of determinants is **not** required, students should be able to use the relevant formula to write down the multiplicative inverse of a 2 × 2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ where } ad - bc \neq 0$$

They should also be able to verify that, for an invertible square matrix **A**:

$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$

where **I** is the relevant $n \times n$ identity matrix for multiplication. They should be familiar with the relationship:

$\mathbf{AB} = k\mathbf{I}$ implies that $\frac{1}{k}\mathbf{B} = \mathbf{A}^{-1}$ where k is a non-zero constant and its application to finding the inverse of a given matrix. Students are **not** required to be familiar with the method of gaussian eliminations, or methods based on determinants, for the solution of systems of simultaneous linear equations.

Example 1

A fast food shop sells meals in three sizes, small, regular and large. Over a three day trading period it sells the following numbers of each sized meal with the given daily taking:

Day	**Number of meals sold**			**Daily taking**
	small	medium	large	
Friday	35	52	46	$601.85
Saturday	60	79	106	$1133.85
Sunday	47	74	51	$768.30

If x = price of small sized meal, y = price of medium sized meal and z = price of large sized meal respectively, then this data can be represented in matrix form as:

$$\mathbf{X}\text{ (price)} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathbf{S}\text{ (sales)} = \begin{bmatrix} 35 & 52 & 46 \\ 60 & 79 & 106 \\ 47 & 74 & 51 \end{bmatrix} \quad \mathbf{T}\text{ (daily takings)} = \begin{bmatrix} 601.85 \\ 1133.85 \\ 768.30 \end{bmatrix}$$

and the price of each type of meal can be determined simultaneously by solution of the matrix equation **SX** = **T** for **X:**

$$\begin{bmatrix} 35 & 52 & 46 \\ 60 & 79 & 106 \\ 47 & 74 & 51 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 601.85 \\ 1133.85 \\ 768.30 \end{bmatrix}$$

where $\mathbf{X} = \mathbf{S}^{-1}\mathbf{T}$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 27.446\ldots & -5.410\ldots & -13.510\ldots \\ -13.827\ldots & 2.712 & 6.834\ldots \\ -5.230\ldots & 1.050\ldots & 2.553\ldots \end{bmatrix}\begin{bmatrix} 601.85 \\ 1133.85 \\ 768.30 \end{bmatrix} = \begin{bmatrix} 3.85 \\ 4.25 \\ 5.35 \end{bmatrix}$$

That is, the price of a small meal is $3.85, a medium sized meal $4.24, and a large meal $5.35.

It should be noted that as students are expected to use technology to carry out the relevant computations, the inverse matrix $\mathbf{S}^{-1}$ need not be displayed explicitly, although it will likely be referred to in describing the solution process.

Transition matrices

A transition matrix is used to determine the state of a system that undergoes a series of discrete changes, or *transitions*. The likelihood or proportion associated with the system being in a given state depends directly on that of the immediately preceding state, and remains constant from one transition to the next.

Students will be required to determine the transition matrix **T** and initial state matrix (vector) $\mathbf{S_0}$ from given information, determine subsequent state matrices, $\mathbf{S}_n$, and be able to identify a steady state as one in which there is no noticeable change from one state matrix to the next, that is where $\mathbf{T} \times \mathbf{S}_n \approx \mathbf{S}_n$ (this will occur if $\mathbf{T}^2$ has no zero elements, even if **T** has some zero elements).

Example 1

A group of 400 students were surveyed regarding their daily lunchtime drink, and their choice of lunchtime drink was monitored over several days. They were asked to nominate the lunchtime drink they had consumed on a particular day. The results were:

Type of lunchtime drink	Number of students from the group
Mineral Water	136
Fruit Juice	64
Lemonade	112
No drink or some other kind of drink	88

It is known that half the students who drink mineral water for lunch on a given day also drink it on the following day. Ten per cent change to a fruit juice and 15% change to lemonade. The remainder have no lunchtime drink on the next day.

Seventy per cent of the students who drink fruit juice for lunch on a given day also drink fruit juice on the following day, 15% switch to mineral water, 10% switch to lemonade and 5% have some other form of lunchtime drink or none at all.

Fifty per cent of lemonade drinkers on a given day drink lemonade the next day, 27% change to mineral water and 12% change to fruit juice. The remainder choose another form of lunchtime drink, or none at all.

Sixteen per cent of students who did not have a drink (or had other kind of drink) for lunch on a given day choose a mineral water on the subsequent day. Forty four per cent do not drink any lunchtime drink on the next day, and the rest are equally divided between drinking lemonade and fruit juice.

Assuming that student choice of lunchtime drink depended only on the choice of drink on the previous day, determine the numbers, and proportions, of drinkers of each type of lunchtime drink four days after the initial survey was taken.

This situation can be modelled using transition matrices where:

$$\mathbf{S}_0 = \begin{bmatrix} 136 \\ 64 \\ 112 \\ 88 \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} 0.5 & 0.15 & 0.27 & 0.16 \\ 0.1 & 0.7 & 0.12 & 0.2 \\ 0.15 & 0.1 & 0.5 & 0.2 \\ 0.25 & 0.05 & 0.11 & 0.44 \end{bmatrix} \text{ and } \mathbf{S}_{n+1} = \mathbf{T}\times\mathbf{S}_n$$

then:

$$\mathbf{S}_1 = \mathbf{T} \times \mathbf{S}_0$$
$$\mathbf{S}_2 = \mathbf{T} \times \mathbf{S}_1 = \mathbf{T} \times \mathbf{T} \times \mathbf{S}_0 = \mathbf{T}^2 \times \mathbf{S}_0$$
$$\mathbf{S}_3 = \mathbf{T} \times \mathbf{S}_2 = \mathbf{T} \times \mathbf{T} \times \mathbf{T} \times \mathbf{S}_0 = \mathbf{T}^3 \times \mathbf{S}_0$$

and the state after four days, $\mathbf{S}_4$ is found by computing

$$\mathbf{S}_4 = \mathbf{T} \times \mathbf{S}_3 = \mathbf{T} \times \mathbf{T} \times \mathbf{T} \times \mathbf{T} \times \mathbf{S}_0 = \mathbf{T}^4 \times \mathbf{S}_0$$

$$= \begin{bmatrix} 0.5 & 0.15 & 0.27 & 0.16 \\ 0.1 & 0.7 & 0.12 & 0.2 \\ 0.15 & 0.1 & 0.5 & 0.2 \\ 0.25 & 0.05 & 0.11 & 0.44 \end{bmatrix}^4 \begin{bmatrix} 136 \\ 64 \\ 112 \\ 88 \end{bmatrix}$$

$$\approx \begin{bmatrix} 111.35 \\ 117.89 \\ 90.74 \\ 80.02 \end{bmatrix}$$

Thus, the numbers, and proportions of students drinking each type of lunchtime drink on the four days after the initial survey are:

Type of lunch-time drink	Number of students from the group (rounded to the nearest person)	Proportion of the group (as a percentage)
Mineral Water	111	27.75
Fruit Juice	118	29.50
Lemonade	91	22.75
No drink or some other kind of drink	80	20.00

Note that the sum of values across categories is constant, subject to some small rounding error.

A steady state is indicated when $\mathbf{T} \times \mathbf{S}_{steady} = \mathbf{S}_{steady}$, or practically, when $\mathbf{T} \times \mathbf{S}_n \approx \mathbf{S}_n$ for sufficiently large values of n, that is, no noticeable change from one state matrix to the next for larger values of n. For this example, this occurs, correct to three decimal places, when $n = 16$:

$$\mathbf{T}^{16} \approx \begin{bmatrix} .275 & .275 & .275 & .275 \\ .310 & .310 & .310 & .310 \\ .222 & .222 & .222 & .222 \\ .194 & .194 & .194 & .194 \end{bmatrix}$$

and the corresponding state matrix is:

$$\mathbf{S}_{16} = \begin{bmatrix} 108 \\ 124 \\ 88 \\ 76 \end{bmatrix}$$

That is, the system reaches a steady state at proportions of approximately 0.275, 0.310, 0.222, 0.194 (with rounding error at the fourth decimal place) for mineral water, fruit juice, lemonade, and other (or no) lunchtime drink respectively, with the corresponding numbers of drinkers, rounded to the nearest person, in each category: 110, 124, 89 and 78. Note that proportions of the group of students could alternatively have been used to specify the initial state matrix. In this case, the subsequent state matrices would also represent proportions.

Some possible contexts for coursework assessment – analysis task

The following could be used as suitable contexts for developing coursework assessment (analysis) tasks for the *Matrices* module:

- stock inventory/sales applications
- use of transition matrices to model changes in populations; weather patterns or competition between two rival teams
- applications to networks (graph theory) in particular connectedness of graphs
- strategies in elementary game theory
- matrix coding and decoding of messages
- matrix row operations and applications to gaussian elimination and the determination of inverse matrices
- iterative approaches to numerical computation using matrices, such as powers of the matrix.

$\begin{pmatrix} 1 & n \\ 1 & 1 \end{pmatrix}$ to compute rational *approximations* to $\sqrt{n}$ given an initial rational approximation $\dfrac{a}{b}$ represented by the matrix $\begin{pmatrix} a \\ b \end{pmatrix}$; or powers of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ to compute sub-sequences of the Fibonacci sequence f of the form $\{f_{n-1}, f_n, f_{n+1}\}$.

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