# VCE MATHEMATICS 2000 Application task CD ROM Resource

## Background

Cambridge University Press, in partnership with the Victorian Board of Studies, is producing a CD ROM teacher resource to support teachers in their implementation of coursework assessment in mathematics. This resource will provide illustrative samples of how application tasks could be devised with respect to a variety of themes for each of Further Mathematics Unit 3, Mathematical Methods Unit 3 and Specialist Mathematics Unit 4. Samples of similar tasks appropriate for General Mathematics Units 1 and 2 and Mathematical Methods Units 1 and 2 will also be included in the CD ROM teacher resource.

The following material provides a *draft* sample task for each of Further Mathematics Unit 3, Mathematical Methods Unit 3 and Specialist Mathematics Unit 4, with possible solution notes. It should be noted that this material is illustrative only, and that teachers will need to take into account a range of factors in locally devising suitable tasks for their students (for example, the background coverage of content from the various areas of study for the unit, possible uses of technology and the intended time on task).

The revised VCE 2000 Assessment Guide for Mathematics provides advice on the nature and scope of the application task for each of Further Mathematics, Mathematical Methods and Specialist Mathematics.

The CD ROM teacher resource will also illustrate how several different application tasks could be devised with respect to a particular theme and context. The resource provides advice on the processes which could be used to devise suitable tasks for this component of coursework assessment in mathematics, based on starting points generated from previous unit 3 and 4 mathematics tasks.

Material on the CD ROM teacher resource will be presented in a format in which material can be readily adapted by teachers, and will also contain write enabled versions of previous VCE mathematics tasks from the period 1989 – 99. Other sample material from the CD ROM teacher resource will be available for inspection from Cambridge University Press at the Mathematical Association of Victoria (MAV) Annual Conference at Monash University, 2nd and 3rd December 1999.

For further details about the CD ROM teacher resource, please contact Peter Cribb, Cambridge University Press, telephone (03) 9568 0322 or email pcribb@cup.edu.a

## FURTHER MATHEMATICS

This sample **Data analysis** application task illustrates how a particular context***, 'trends in the number of births in Victoria over several years'***, could be developed from data related to a theme such as **'Characteristics of a Population'**. This task involves times series analysis with a seasonal component. The data for this task can be found in Table 2 of the Further Mathematics 1998 Investigative project booklet.

### Introduction

The number of births registered in Victoria changes over time. Governments maintain statistics on this variable for planning purposes. The following investigation looks at whether or not the number of births in Victoria could be considered to contain a seasonal component, and also if there is any trend in this variable, which could be modeled with some degree of confidence.

### Part 1

Table 2 gives the number of births in Victoria, quarterly from March 1978.

Choose three consecutive years of birth data starting from a randomly selected March quarter from March 1979 to March 1993 inclusive (explain your method of selection).

Use suitable univariate analysis techniques to summarise and describe the distribution of birth rates in you sample data. For your selected period, construct a time series plot and comment on its features.

### Part 2

The data from the three years you have selected can be used to calculate 'seasonal indices' for the birth rate. One for doing this is as follows:

**Step 1**

Calculate the average quarterly number of births for each year separately, by summing the quarterly figures for the year and dividing by 4.

**Step 2**

Use the average figure to determine the proportion of the average number of births that were recorded each quarter. That is, if the quarterly average for a particular year is 15000 births, and the actual number for the March quarter is 16500 births, then the proportion of an average quarter recorded for March is 16500/15000 = 1.10.

**Step 3**

Determine a seasonal index by averaging these proportions for each quarter over the three years. Briefly discuss what these 'seasonal indices' suggest about the seasonal pattern of the number of births in Victoria. For your selected period, use the seasonal indices to deseasonalise your data.

Use this deseasonalised data to produce a scatterplot of the number of births as a function of time, and comment on the scatterplot.

**Part 3**

Determine the equation of the least squares regression line for the deseasonalised and draw the corresponding graph on your scatterplot. Interpret the vertical axis intercept and slope of the regression line in terms of number of births and time.

Use your regression model to predict the number of births for each quarter of the next year outside the range of your sample (you will need to re-seasonalise this prediction by multiplying by the appropriate seasonal indices to obtain seasonalised predictions) and compare them with the actual figures.

Use percentage error as a measure of accuracy, to comment on the differences between estimated and actual values and the reliability of your model.

## Solution and Comments

### Aim

To investigate the number of births in Victoria over a period of time, with a view to developing a mathematical model for number of births which may be used for prediction.

## Part 1

### Selection of data

Teachers may request students to make a random selection or allocate a set of data. The purpose is to ensure that each student is working on a unique set of data. Three consecutive years must be randomly selected from the data, starting between March 1979 and March 1993. There are 15 years beginning with a March quarter. Numbering each possible March quarter from 1 to 15 a random selection is made, giving the following data. The quarters have been allocated a code, with March 1990 coded as Quarter 1 and so on.

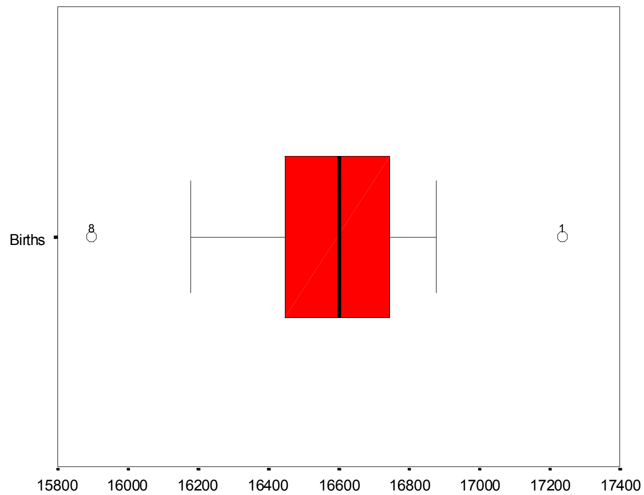
| Quarter | Coded Time | Births |
|---|---|---|
| Mar 1990 | 1 | 17237 |
| June 1990 | 2 | 16875 |
| Sept 1990 | 3 | 16517 |
| Dec 1990 | 4 | 16529 |
| Mar 1991 | 5 | 16597 |
| June 1991 | 6 | 16626 |
| Sept 1991 | 7 | 16604 |
| Dec 1991 | 8 | 15897 |
| Mar 1992 | 9 | 16798 |
| June 1992 | 10 | 16179 |
| Sept 1992 | 11 | 16695 |
| Dec 1992 | 12 | 16378 |

### Univariate analysis

The number of births is a numeric variable. To describe the distribution of the numeric variables students should use one graphical display (histogram, stemplot, or boxplot) and summary statistics. These should then be described and interpreted in terms of the variable in the data set.

**Variable: Births**

Boxplot



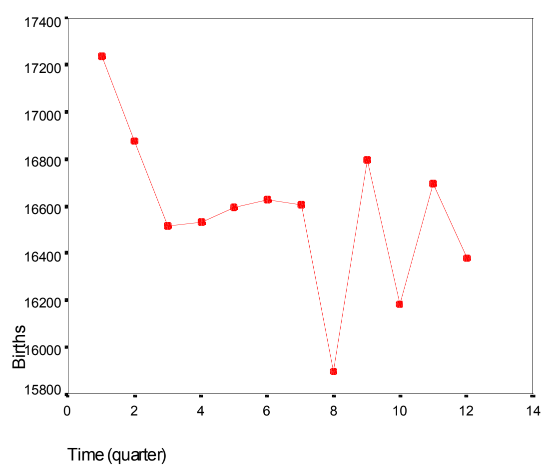

Summary Statistics

| Mean | St Dev | Min | Q1 | Med | Q3 | Max | IQR |
|---|---|---|---|---|---|---|---|
| 16577.8 | 388.0 | 15897 | 16412.8 | 16600.5 | 16772.3 | 17237 | 359.5 |

The distribution of number of births in this sample is slightly negatively skew, with two outliers in Quarter number 1 (March 1990), when the number of births was particularly high, and Quarter number 8 (December 1991) when the number of births was particularly low. The typical number of births in a quarter is around 16600, and in about 95% of quarters the number of births was between 15802 and 17354.

**Time series analysis**

A time series plot of number of births against quarter will illustrate the pattern in the data, and in particular if any trend is apparent.




From the time series it can be seen that the number of births drops from Quarter 1 (March 1990) and then remains reasonably constant over the next few quarters.

However, more variation is seen in the number of births over the last five time periods. There does not appear from this plot to be any trend in the number of births, nor any clear indication of seasonality.

## Part 2

**Determination of Seasonal Indices**

*There are many acceptable methods for calculating seasonal indices, and teachers should use their preferred method here.*

Seasonal indices can be calculated as follows:

1. Add the quarterly births for each year and divide the result by 4 giving the quarterly average number of births each year:

| Quarter | 1 | 2 | 3 | 4 | Total | Average |
|---|---|---|---|---|---|---|
| Births | 17237 | 16875 | 16517 | 16529 | 67158 | 16790 |

2. The **actual** number of births for each quarter is divided by the quarterly average for that year, resulting in a seasonal index for the quarter. The following table shows these calculations for each period in the sample.

| Year | Quarter | Calculation | Seasonal Index |
|---|---|---|---|
| 1990 | Mar | 17237/16789 | 1.0267 |
| | June | 16875/16789 | 1.0051 |
| | Sept | 16517/16789 | 0.9838 |
| | Dec | 16529/16789 | 0.9845 |
| 1991 | Mar | 16597/16431 | 1.0101 |
| | June | 16626/16431 | 1.0119 |
| | Sept | 16604/16431 | 1.0105 |
| | Dec | 15897/16431 | 0.9675 |
| 1992 | Mar | 16798/16512 | 1.0173 |
| | June | 16179/16512 | 0.9798 |
| | Sept | 16695/16512 | 1.0110 |
| | Dec | 16378/16512 | 0.9918 |

3. The seasonal indices for each quarter are then averaged over the three years to produce a final seasonal index, which is based on all the available information.

| Year | Quarter 1 | Quarter 2 | Quarter 3 | Quarter 4 |
|---|---|---|---|---|
| 1990 | 1.027 | 1.005 | 0.984 | 0.985 |
| 1991 | 1.010 | 1.012 | 1.011 | 0.968 |
| 1992 | 1.017 | 0.980 | 1.011 | 1.011 |
| Seasonal Index | 1.02 | 1.00 | 1.00 | 0.98 |

**Interpretation of Seasonal Indices**

When the resulting index is greater than 1, the number of births in that quarter is said to be above average and if the index is below 1, the number of births in that quarter is said to be below average. It can be seen from the values calculated that the two middle quarters, June and September, record an average number of births. In the March quarter the seasonal index is 1.02, indicating that in this quarter the number of births is about 2% above the average quarter. In the December quarter the seasonal index is 0.98, indicating that in this quarter the number of births is about 2% below the average quarter.

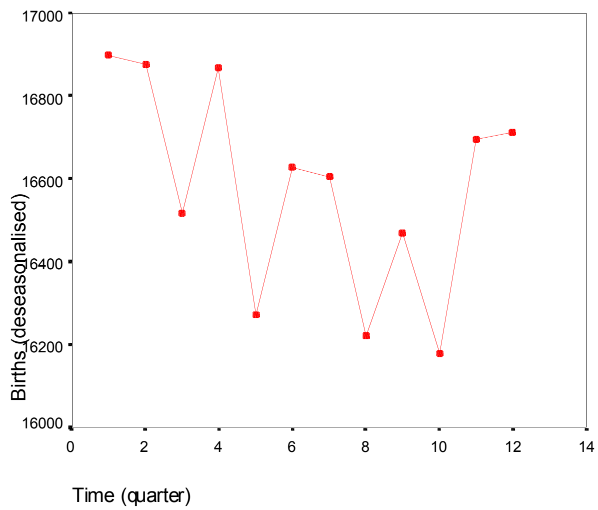
**Deseasonalisation of the data**

To deseasonalise the data the actual value is divided by the corresponding seasonal index in each quarter:

Deseasonalised value = actual value/ seasonal index

| Quarter | Coded Time | Births (Actual) | Calculation | Births (Deseasonalised) |
|---|---|---|---|---|
| Mar 1990 | 1 | 17237 | 17237/1.02 | 16899 |
| June 1990 | 2 | 16875 | 16875/1.00 | 16875 |
| Sept 1990 | 3 | 16517 | 16517/1.00 | 16517 |
| Dec 1990 | 4 | 16529 | 16529/0.98 | 16866 |
| Mar 1991 | 5 | 16597 | 16597/1.02 | 16272 |
| June 1991 | 6 | 16626 | 16626/1.00 | 16626 |
| Sept 1991 | 7 | 16604 | 16604/1.00 | 16604 |
| Dec 1991 | 8 | 15897 | 15897/0.98 | 16221 |
| Mar 1992 | 9 | 16798 | 16798/1.02 | 16469 |
| June 1992 | 10 | 16179 | 16179/1.00 | 16179 |
| Sept 1992 | 11 | 16695 | 16695/1.00 | 16695 |
| Dec 1992 | 12 | 16378 | 16378/0.98 | 16712 |

Using the deseasonalised data another time series plot of births against time can be constructed.




The time series plot of the deseasonalised data shows a general decreasing trend over most of the three-year period in question.

## Part 3

### Fitting the regression line

Since the number of births seems to be decreasing over the three-year period a straight line is an appropriate model for this trend.

A straight line has the form:

$$y = a + bx$$

where a is the intercept of the line with the y-axis, and b is the slope.

The equation of this line can be found using least squares regression, on a calculator or in an appropriate computer package.

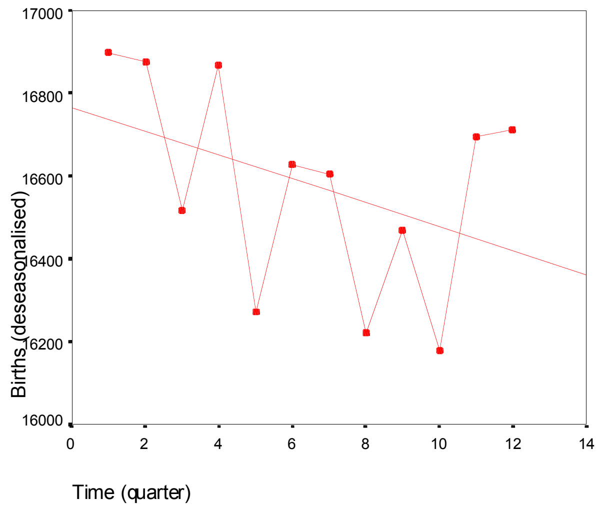
Here:

$$a = 16764$$
$$b = -28.7$$

Thus, the model for the number of births over time is:

$$\text{Number of births} = 16764 - 28.7 \times \text{time period}$$

where time period is the coded time of a particular quarter, and is shown in the following plot.




The y-intercept indicates that the number of births for the period prior to quarter 1 in this sample is predicted to be 16764. This value is not of any particular interest.

The gradient of –28.7 indicates that over the period of our sample, the number of births is decreasing by an average of 29 births per quarter.

**Predicting the number of births**

The model will be used to predict the number of births for the following year after the sample, time periods 13, 14, 15 and 16 according to the coding.

For example, for time period 13:

$$\text{Predicted deseasonalised births} = 16764 - 28.7 \times 13 = 16391$$

The model calculates the deseasonalised births, so the values determined need to be re-seasonalised to allow comparison to the original data.

The method is:

$$\text{Predicted births} = \text{predicted deseasonalised births} \times \text{seasonal index}$$

These calculations are shown in the following table:

| Quarter | Coded time | Predicted deseasonalised births | Predicted births |
|---|---|---|---|
| March 1993 | 13 | 16391 | 16719 |
| June 1993 | 14 | 16362 | 16362 |
| September 1993 | 15 | 16334 | 16334 |
| December 1993 | 16 | 16305 | 15979 |

To evaluate the fit of the model, the predicted numbers of births needs to be compared to the actual numbers of births. A measure of the goodness of fit can be given by calculating for each quarter the percentage error for each of the predictions.

That is:

$$\% \text{ Error} = \frac{\text{Predicted Births} - \text{Actual Births}}{\text{Actual Births}} \times 100$$

These values are given in the following table:

| Quarter | Predicted Births | Actual Births | Error | % Error |
|---|---|---|---|---|
| Mar1993 | 16719 | 16552 | 167 | 1.0 |
| Jun 1993 | 16362 | 15917 | 445 | 2.8 |
| Sep1993 | 16334 | 16240 | 94 | 0.6 |
| Dec 1993 | 15979 | 15555 | 424 | 2.7 |

The percentage errors are all positive, indicating that the model is overestimating the number of births. However, the percentage errors are small, meaning that the predictions based on this model are quite good for the period. Of course, there are many factors that affect the number of births, so one would not expect a prediction based on only one variable to be more accurate that this. We would not expect predictions for time periods further away from the sample to be as accurate.

**Conclusion**

By fitting a trend line to the deseasonalised birth data, a mathematical model was found that allowed for reasonable predictions to be made of the birth numbers within one year of the sample time. The model demonstrated that the birth rate was decreasing on average by 29 births each quarter, although this cannot continue indefinitely as eventually the birth rate would become zero. The model is satisfactory for this time frame, but it would become a poor predictor as time progressed.

## MATHEMATICAL METHODS

This sample **function and calculus** application task illustrates how a particular context, ***'investigating the behaviour of functions with rules of the form: $f(x) = (a-x)^m(b-x)^n$*** could be developed in a relation to a theme such as ***'Describing curves'***. This task involves the domain and range of functions, polynomial functions, and differential calculus.

**Part 1**

Initially we consider the family of functions with rules of the form $f(x) = x^m(2-x)^n$
Where $m$ and $n$ are positive integers. First consider the case where $m$ and $n$ are both equal to one. Sketch this function for a suitable domain, determining any turning points.

Now consider the case where $m$ and $n$ are integers between 1 and 3 inclusive. Systematically vary the values of $m$ and $n$ to determine the effect which each has on the behaviour of the function, in particular paying attention to:

- the number of stationary points
- location of stationary points
- nature of stationary points

Use appropriate technology to produce graphs to illustrate your findings, carefully labeling all key features of the graph. Consider the points of intersection of the curves, which are produced.

**Part 2**

Now consider the more general case where $m$ and $n$ are positive integers for the family of functions with rules of the form $f(x) = x^m(2-x)^n$ . Draw a careful selection of graphs which illustrate the behaviour of the functions as $m$ and $n$ are varied.

Consider the functions formed by varying the parity (odd or even) of $m$ and $n$ and generalise your results from **Part 1**.

**Part 3**

Now consider the particular family of curves of the form $f(x)=(a-x)^m(b-x)^n$ , where $a$ and $b$ are positive real numbers, generated in each of the following cases:

**Case 1 -** $m$ and $n$ are both even **Case 2 -** $m$ is even and $n$ is odd

Give a general description for the curves produced in each case.

## Describing Curves Solutions

### Part 1

For $f(x) = x^n(2 - x)^m$

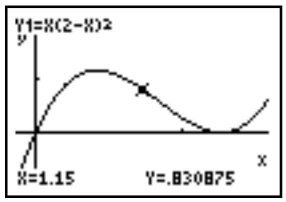
If $n = m = 1$, then the graph of $y = f(x)$ is a parabola with vertex at (1,1) and axes intercepts (0,0) and (2,0). The parabola has a local maximum at (1,1)

Now consider different values of $m,n$ integers, with $1 \le m \le 3$ and $1 \le n \le 3$

***n* = 1 and *m* = 2**

Axes intercepts at (0,0) (2,0) ; local maximum at $(\frac{2}{3}, \frac{32}{27})$ and local minimum at (2,0).

$y = f(x)$ is a cubic function with coefficient of $x^3$ negative.

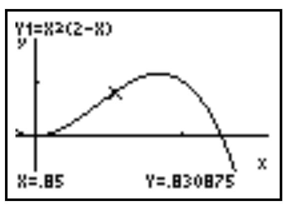



Window settings: Xmin = –0.2,Xmax = 2.5, Ymin = –1, Ymax = 2.2

***n* = 2 and *m* = 1**

Axes intercepts at (0,0) (2,0) ; local maximum at $(\frac{4}{3}, \frac{32}{27})$ and local minimum at (0,0).

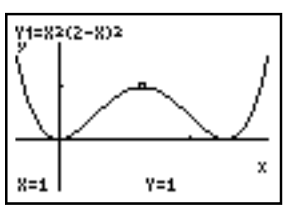
The graph of $y = x(2 - x)^2$ is the reflection of $y = x^2(2 - x)$ in the line $x = 1$




Window settings: Xmin = –0.2, Xmax = 2.5, Ymin = –1, Ymax = 2.2

***n* = 2 and *m* = 2**

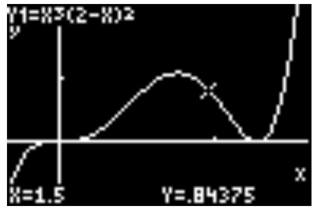
Axes intercepts at (0,0) (2,0) ; local maximum at (1,1), local minimums at (0,0) and (2,0)




Window settings: Xmin = –0.5, Xmax = 2.5, Ymin = –1, Ymax = 2.2

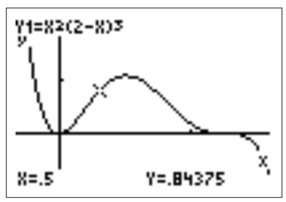
***n* = 3 and *m* = 2**

Axes intercepts at (0,0) (2,0) ; local minimum at (2,0), local maximum at ($\frac{6}{5}$ , $\frac{3456}{3125}$ ) and a stationary point of inflexion at (0,0).




Window settings: Xmin = –0.5, Xmax = 2.5, Ymin = –1, Ymax = 2.2

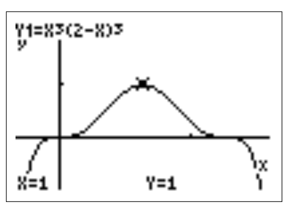
The graph of $y = x^2 (2 - x)^3$ is a reflection of the graph of $y = x^3 (2 - x)^2$ in the line $x = 1$




Window settings: Xmin = –0.5, Xmax = 2.5, Ymin = –1, Ymax = 2.2

***n* = 3 and *m* = 3**

Stationary points of inflexion at (0,0) and (2,0) and a local maximum at (1,1)




Window settings: Xmin = –0.5, Xmax = 2.5, Ymin = –1, Ymax = 2.2

**Part 2**

For $f(x) = x^n (2 - x)^m$

Stationary points occur for $x = 0$, $x = 2$ and $x = \frac{2n}{n+m}$

Coordinates (0,0) (2,0) $\left(\frac{2n}{n+m} , \left(\frac{2n}{n+m}\right)^n \left(\frac{2m}{n+m}\right)^m \right)$

There is a local maximum for $x = \frac{2n}{n+m}$ for all values of $n$ and $m$

The nature of the stationary points for (0,0) and (2,2) depends on the parity of $n$ and $m$

***n* even and *m* even**
local minimum at (0,0) and (2,0)

***n* odd and *m* even    $n > 1$**
local minimum at (2,0) and stationary point of inflexion at (0,0)

***n* even and *m* odd    $m > 1$**
local minimum at (0,0) and stationary point of inflexion at (2,0)

***n* odd and *m* odd    $n, m > 1$**
stationary points of inflexion at (0,0) and (2,0)

It should also be shown that the curves intersect at the (1,1)

**Part 3**

$y = f(x) = (a - x)^m (b - x)^n$    $a$ not equal to $b$

**In general**

Axes intercepts at $x = a$ and $x = b$

Stationary points occur at ($a$, 0) ($b$, 0) and $\left(\frac{nb + ma}{n + m}, f(\ )\right)$

for $n > 1$ and $m > 1$

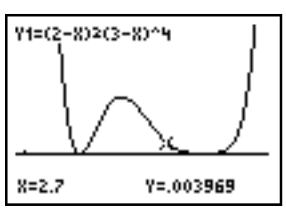
The reflection of the graph of $y = (a - x)^n (b - x)^m$ in the line $x = \frac{a + b}{2}$ is

$y = (b - x)^n (a - x)^m (-1)^{n+m}$

There are two cases to consider here.

***m* even and *n* even**

$f(x) > 0$ for $x$ e R\{$a, b$}
local minimums at ($a$, 0) and ($b$, 0)
local maximum at $\left(\frac{nb + ma}{n + m}, f(\ )\right)$
If $m = n$, local maximum at $\left(\frac{b + a}{2}, f(\ )\right)$
The graph below is for $m = 2$ and $n = 3$ with $a = 2$ and $b = 3$ . The $y$ axis intercept is not shown but is at (0 , 324).

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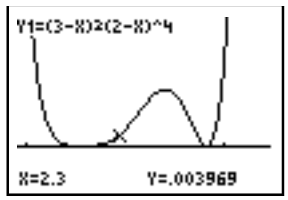
For $m$ and $n$ even $y = (a-x)^n (b-x)^m$ is the reflection of

$y = (b-x)^n (a-x)^m$ in the line $x = \frac{a+b}{2}$

The graph below is for $m = 2$ and $n = 4$ with $b = 2$ and $a = 3$ . The $y$ axis intercept is not shown but is at (0 , 144).

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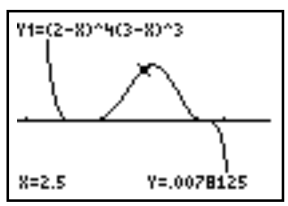
***m* even and *n* odd (assume *b* >*a* )**

local maximum at $\left(\frac{nb+ma}{n+m}, f(\ )\right)$ local minimum at ($a$, 0) and stationary point of inflexion at ($b$, 0)

The graph below is for $m = 4$ and $n = 3$ with $a = 2$ and $b = 3$ . The $y$ axis intercept is not shown but is at (0 , 432).

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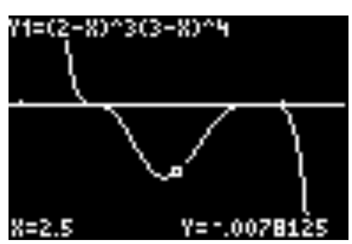
Reflection of $y = (a-x)^m (b-x)^n$ in the line $x = \frac{a+b}{2}$ is $y = -(b-x)^m (a-x)^n$

Therefore for $m$ odd and $n$ even, local minimum at $\left(\frac{nb+ma}{n+m}, f(\ )\right)$, stationary point of inflexion at ($a$, 0) and local maximum at ($b$, 0)

The graph below is for $m = 3$and $n = 4$ with $a = 2$ and $b = 3$ . The $y$ axis intercept is not shown but is at (0 , 648).



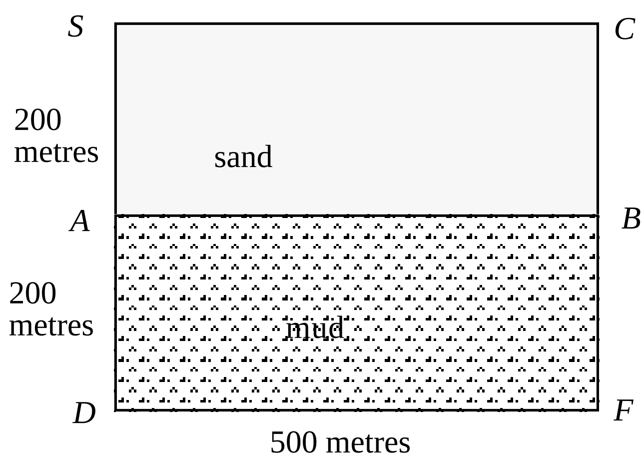

Window settings: Xmin = 1.5, Xmax = 3.5, Ymin = –0.015, Ymax = 0.01

## SPECIALIST MATHEMATICS

This sample **problem** application task illustrates how a particular context***, 'the game show race Gladiatrix',*** could be developed in relation to a theme such as **'Finding the best path'**. This task involves the domain and range of functions, circular functions and their inverses, calculus and an iterative approach to finding approximate solutions to equations.

### Part 1

In a weekly game show contest 'Gladiatrix', two contestants race over identical courses to determine who can run from the start to the finish in the shortest time. Each course is set out as in the figure below. *SABC* and *ADFB* are rectangular boundaries, each with dimensions 200 metres by 500 metres. The interior of *SABC* is filled with fine sand and the interior of *ADFB* is filled with thick mud. *S* is the starting point and *F* is the finishing point.

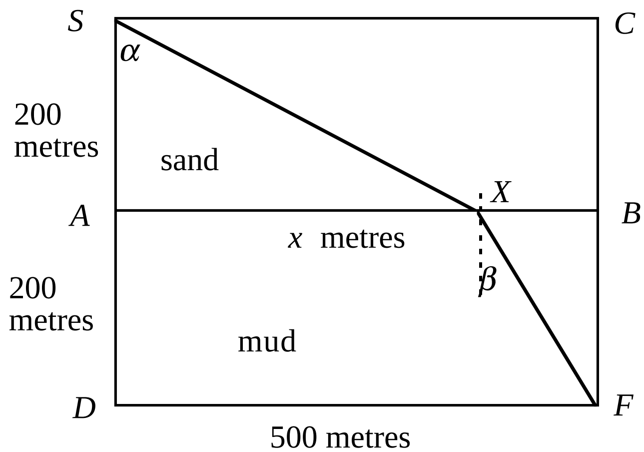



One of the contestants, Kylie, can run at 4.5 metres per second across the sand but at only 2.5 metres per second on the muddy surface. Suppose Kylie uses the direct route from *S* to *F*.

How far will she run and how long will it take her?

Now let us explore what happens if Kylie takes other routes across the two surfaces.

Let $X$ be a point on $AB$ and let the distance from $A$ to $X$ be $x$ metres. Suppose that Kylie goes from $S$ directly to $X$ and then from $X$ directly to $F$. Let $d$ metres be the distance that Kylie runs using this route and let $t$ seconds be the time that she takes. In the figure below, the route from $S$ to $X$ followed by $X$ to $F$ is drawn in and angles $\alpha$ and $\beta$ are defined as shown.




Show that $d = 200(\sec\alpha + \sec\beta)$ and prove that $\alpha$ and $\beta$ are also related by the equation

$$\tan\alpha + \tan\beta = k \qquad \text{… equation A}$$

where $k$ is a constant.

Find an expression for $d$ in terms of $\alpha$ and state the implied domain of the corresponding function. Find the total distance Kylie runs if she starts off at an angle of 50°, and state the angle $\beta$ she must use when she reaches the second surface.

## Part 2

Find an expression for $t$ , the time taken for her to travel from $S$ to $F$, in terms of $\alpha$, and state the implied domain of the corresponding function. Use a graphics calculator or computer graphing software to plot the graph of $t$ against $x$ and comment on what this graph shows.

Use your graph to find the minimum value of $t$ and the corresponding value of $\alpha$. What angle $\beta$ must she use when she reaches the second surface? State, to the nearest metre, the distance Kylie runs and the time, to the nearest second, that she takes if she uses the route that minimises $t$.

Emily, Kylie's Gladiatrix opponent, can run at 4 metres per second across the sand and 3 metres per second on the muddy surface. Use the same approach as to determine Emily's best route. Who wins the contest and by how many seconds?

## Part 3

Suppose we consider the problem of finding the best route for Kylie using the following alternative approach.

Let $t_1$ be the time in seconds that Kylie takes from *S* to *X* and let $t_2$ be the time in seconds that she takes from *X* to *F*, so that her total time is $t = t_1 + t_2$.

Write down expressions for $t_1$ and $t_2$ in terms of *x*, and find $\frac{dt_1}{dx}$ and $\frac{dt_2}{dx}$ in terms of *x*.

With the angles $\alpha$ and $\beta$ as shown in the original figure above, express $\frac{dt_1}{dx}$ and $\frac{dt_2}{dx}$ in terms of $\alpha$ and $\beta$ respectively.

**Hence** express $\frac{dt}{dx}$ in terms of $\alpha$ and $\beta$ and show that the minimum time occurs when

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \qquad \text{… equation B}$$

where *a* and *b* are constants.

Give an interpretation of these constants in terms of the situation being modeled.

To find the values of $\alpha$ and $\beta$ corresponding to the minimum time, one approach is to solve the equations A and B simultaneously using an iterative procedure. [You are not expected to justify the iterative procedure, which follows, just to use it.]

**Step 1**

Start with an initial guess for $\alpha$ such as 45 degrees, and use equation A to find the corresponding value of $\beta$. With this value of $\beta$, use equation B to find the corresponding value of $\alpha$. Now use this new value of $\alpha$ in equation A to find the next value of $\beta$ and so on until sufficiently accurate values of $\alpha$ and $\beta$ are obtained which satisfy equations A and B simultaneously.

**Hence** verify that the best route for Kylie is the same as that found previously.

Suppose that in the iterative procedure described above, we start with an initial guess for $\beta$ of 30degrees and then proceed as before. Follow the iterative process in this case and comment on what happens.

Suppose another contestant can run at *u* metres per second across the sand and *v* metres per second on the muddy surface. Determine the equations that need to be solved simultaneously in this case in order to find the values of $\alpha$ and $\beta$ corresponding to the minimum time. Explain how you would set up an iterative procedure to solve these equations for given values of *u* and *v*.

**Solutions to Gladiatrix**

## Part 1

Distance *SF* in m $= \sqrt{400^2 + 500^2}$

$= 640.312$

So she will run 640.31 metres.

She runs half this distance across the sand at 4.5 m/s and half across the mud at 2.5 m/s.

Total time in s $= \frac{1}{2} \left[ \frac{640.312}{4.5} + \frac{640.312}{2.5} \right]$

$= 199.21$

So she will take just over 3 min 19 s.

From right-angled triangle *SAX*, *SX* = 200 sec$\alpha$ . Similarly, from right-angled triangle *FBX*, *FX* = 200 sec$\beta$ . So:

$$d = SX + XF$$
$$= 200(\sec\alpha + \sec\beta)$$

From the same triangles, $x = 200\tan\alpha$ and $500 - x = 200\tan\beta$, so adding gives:

$$500 = 200(\tan\alpha + \tan\beta)$$

Dividing through by 200 gives:

$$\tan\alpha + \tan\beta = 2.5 \qquad \text{… equation A}$$

From equation A, $\beta = \mathrm{Tan}^{-1}(2.5 - \tan\alpha)$. Substitution gives:

$$d = 200(\sec\alpha + \sec(\mathrm{Tan}^{-1}(2.5 - \tan\alpha)))$$

Clearly, $\alpha \geq 0$ and $\tan\alpha \leq 2.5$. So the domain is $[0, \mathrm{Tan}^{-1}2.5]$.

[Using $\sec^2\alpha = 1 + \tan^2\alpha$, this expression could also be written in the form:

$$d = 200(\sec\alpha + (7.25 - 5\tan\alpha + \tan^2\alpha)^{1/2})$$

Substituting 50° for $\alpha$ in either expression for $d$, we get $d$ = 640.478 (m). Also, $\alpha$ = 52.61°.

**Part 2**

$$t = \frac{SX}{4.5} + \frac{XF}{2.5}$$

$$= 200\left(\frac{\sec\alpha}{4.5} + \frac{\sec\beta}{2.5}\right)$$

$$= 200\left(\frac{\sec\alpha}{4.5} + \frac{\sec(Tan^{-1}(2.5 - \tan\alpha))}{2.5}\right),\ 0 \le \alpha \le \text{Tan}^{-1}2.5$$

Alternatively we could use the form:

$$t = 200\left(\frac{\sec\alpha}{4.5} + \frac{(7.25 - 5\tan\alpha + \tan^2\alpha)^{\frac{1}{2}}}{2.5}\right),\ 0 \le \alpha \le \text{Tan}^{-1}2.5$$

The graphics calculator screen dump in Figure 1 uses a window with dimensions

[0, $\text{Tan}^{-1}2.5$] by [170, 270].

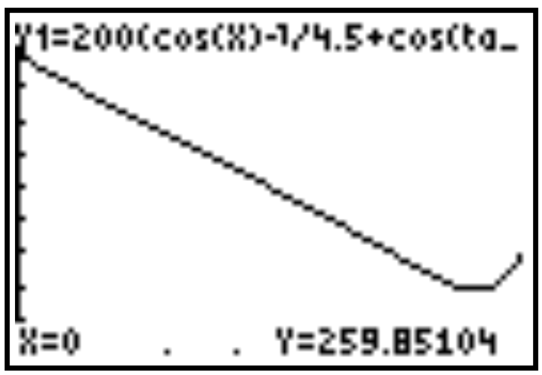



Figure 1

The graph decreases almost linearly until it reaches a minimum near the right-hand end of its domain and then rises. The end-points are at (0, 259.85) and

($\text{Tan}^{-1}2.5$, 199.67). The minimum point on the graph is at (62.64, 188.69) as shown in Figure 2.

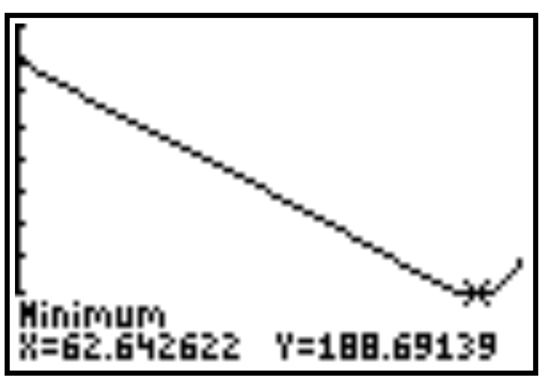


Figure 2

So the minimum value of $t$ is 188.89 (s) and the corresponding value of $\alpha$ is 62.64. Then we find $\alpha = \text{Tan}^{-1}(2.5 - \tan\alpha) = 29.57°$.

Using $d = 200(\sec\alpha + \sec(\text{Tan}^{-1}(2.5 - \tan\alpha)))$, we find $d = 665.16$. So Kylie runs 665 m (to the nearest metre) and takes 189 s (to the nearest second) if she uses the route that minimises $t$.

For Emily, the distance formula is the same function of $\alpha$, and the expression for her time (s) is

$$t = 200\left(\frac{\sec\alpha}{4} + \frac{\sec(Tan^{-1}(2.5 - \tan\alpha))}{3.5}\right),\ 0 \le \alpha \le \text{Tan}^{-1}2.5$$

The graph of $t$ versus $\alpha$ and the minimum point on the graph is shown in Figure 3 (using the same window dimensions as before). So the minimum value of $t$ is 183.96 (s) and the corresponding value of $\alpha$ is 58.96°. Then we find $\beta = \text{Tan}^{-1}(2.5 - \tan\alpha) = 39.98°$.

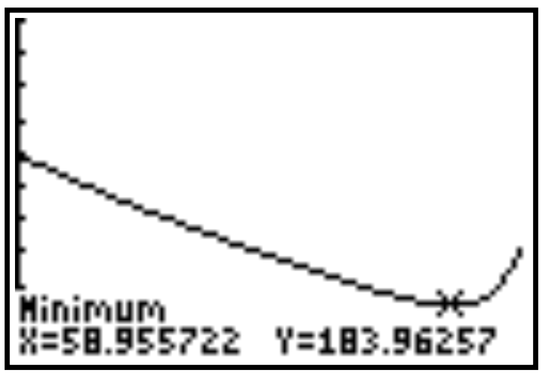



Figure 3

Using $d = 200(\sec\alpha + \sec(\text{Tan}^{-1}(2.5 - \tan\alpha)))$, we find $d = 648.84$. So Emily runs 648 m (to the nearest metre) and takes 184 s (to the nearest second) if she uses the route that minimises $t$. If each contestant uses her own best route, Emily will win the contest by 5 seconds, to the nearest second.

**Part 3**

$$t_1 = \frac{\sqrt{200^2 + x^2}}{4.5}$$

$$t_2 = \frac{\sqrt{200^2 + (500 - x)^2}}{2.5}$$

$$\frac{dt_1}{dx} = \frac{x}{4.5\sqrt{200^2 + x^2}}$$

$$\frac{dt_2}{dx} = \frac{-(500-x)}{2.5\sqrt{200^2+(500-x)^2}}$$

In right-angled triangle *SAX*, $\sin\alpha = \frac{x}{\sqrt{200^2 + x^2}}$

So: $$\frac{dt_1}{dx} = \frac{\sin\alpha}{4.5}$$

In right-angled triangle *FBX*, $\sin\beta = \frac{(500-x)}{\sqrt{200^2+(500-x)^2}}$

So: $$\frac{dt_2}{dx} = -\frac{\sin\beta}{2.5}$$

$t = t_1 + t_2$. So we obtain:

$$\frac{dt}{dx} = \frac{dt_1}{dx} + \frac{dt_2}{dx}$$

$$= \frac{\sin\alpha}{4.5} - \frac{\sin\beta}{2.5}$$

$t$ has a (minimum) stationary point when $\frac{dt}{dx} = 0$, that is when:

$\frac{\sin\alpha}{4.5} = \frac{\sin\beta}{2.5}$ (equation B)

This equation is satisfied when $a = 4.5$ and $b = 2.5$; so $a$ and $b$ are the speeds that Kylie can run across the sand and mud respectively.

If we start with $\alpha = 45°$: equation A gives $\alpha = 23.131°$. Then equation B gives $\alpha = 68.199°$. Substitution into A again gives $\alpha = 31.052°$. Then equation B gives $\alpha = 64.246°$. We continue the process until it stabilises, giving the values $\alpha = 62.64°$ and $\beta = 29.57°$ (correct to one hundredth of a degree). These values are identical to those found previously, so the best route is as found previously.

Note that the iterative procedure lends itself to spreadsheet methods. We have:

$\beta_n = \text{Sin}^{-1}(\frac{5}{9}\ \sin\alpha_{n-1})$ from equation B; then $\alpha_n = \text{Tan}^{-1}(2.5 - \tan\beta_n)$ from equation A.

These may be entered into a spreadsheet as follows: column a contains positive integers $n$; column b contains the values of $\beta_n$ and column c contains the values of $\alpha_n$. Cell c2 has the initial value of α, $\alpha_0 = 45$; cell b3 is then given by =ASIN(5*SIN(C2*PI()/180)/9)*180/PI(); cell c3 is then given by =ATAN(2.5-TAN(B3*PI()/180))*180/PI(); filling down the remaining cells of columns b and c completes the process. The use of PI()/180 and 180/PI() enables the use of degrees rather than radians.

In Figure 4, the result of this iterative procedure shows that the process stabilises in only a few iterations.

| $n$ | $\beta_n$ | $\alpha_n$ |
|---|---|---|
| 0 | | 45 |
| 1 | 23.13114767 | 64.24574502 |
| 2 | 30.02448105 | 62.5134044 |
| 3 | 29.52766813 | 62.65322269 |
| 4 | 29.56878234 | 62.64175397 |
| 5 | 29.56541657 | 62.64269353 |
| 6 | 29.56569235 | 62.64261655 |
| 7 | 29.56566975 | 62.64262285 |
| 8 | 29.5656716 | 62.64262234 |
| 9 | 29.56567145 | 62.64262238 |
| 10 | 29.56567147 | 62.64262237 |

Figure 4

$\alpha$ = 30°: equation B gives $\alpha$ = 64.158°. Then equation A gives $\alpha$ = 23.521°. Substitution into B again gives $\alpha$ = 45.920°. Then equation A gives $\beta$ = 55.726°.

Now when we try to find $\alpha$ from equation B again, we find $\frac{9}{5}$ sin($\beta$) = 1.4874 and $\text{Sin}^{-1}$(1.4874) does not exist, so the process terminates without stabilising. So even though the initial guess for $\beta$ is very close to the actual value found earlier, this iterative scheme does not lead to the solution.

The result using the spreadsheet approach is shown in Figure 5 (note that column b now contains $\alpha$ values and column c $\beta$ values, and the formula for cell b3 uses 9/5 rather than 5/9).

| $n$ | $\alpha_n$ | $\beta_n$ |
|---|---|---|
| 0 | | 30 |
| 1 | 64.15806724 | 23.52148829 |
| 2 | 45.91985044 | 55.72580583 |
| 3 | #NUM! | #NUM! |
| 4 | #NUM! | #NUM! |
| 5 | #NUM! | #NUM! |

Figure 5

$$t_1 = \frac{\sqrt{200^2 + x^2}}{u}$$

$$t_2 = \frac{\sqrt{200^2 + (500 - x)^2}}{v}$$

$$\frac{dt_1}{dx} = \frac{x}{u\sqrt{200^2 + x^2}}$$

$$\frac{dt_2}{dx} = \frac{-(500 - x)}{v\sqrt{200^2 + (500 - x)^2}}$$

In right-angled triangle *SAX*, $\sin\alpha = \dfrac{x}{\sqrt{200^2 + x^2}}$

So: $$\frac{dt_1}{dx} = \frac{\sin a}{u}$$

In right-angled triangle *FBX*, $\sin\beta = \dfrac{(500 - x)}{\sqrt{200^2 + (500 - x)^2}}$

So: $$\frac{dt_2}{dx} = -\frac{\sin b}{v}$$

$t = t_1 + t_2$. So we obtain:

$$\frac{dt}{dx} = \frac{dt_1}{dx} + \frac{dt_2}{dx}$$

$$= \frac{\sin a}{u} - \frac{\sin b}{v}$$

$t$ has a (minimum) stationary point when $\dfrac{dt}{dx} = 0$, that is when $\dfrac{\sin a}{u} = \dfrac{\sin b}{v}$ .

The setting up of a scheme to solve the pair of equations for $\alpha$ and $\beta$ iteratively for given values of $u$ and $v$ needs to consider cases where $u > v$ (as in the above), $u = v$ (in which case iteration is unnecessary) and $u < v$; and the selection of an appropriate initial choice of $\alpha$ or $\beta$ to start the iteration in order to avoid assigning the sin function a value greater than 1 in the iteration process, should be carefully explained.