SPECIALIST MATHEMATICS

Written examination 1

Friday 9 November 2012

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

<table>
<thead>
<tr>
<th>Number of questions</th>
<th>Number of questions to be answered</th>
<th>Number of marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>40</td>
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</tbody>
</table>

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are not permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 9 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your student number in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.
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Instructions

Answer all questions in the spaces provided.
Unless otherwise specified an exact answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.
Take the acceleration due to gravity to have magnitude \( g \) m/s\(^2\), where \( g = 9.8 \).

Question 1
Find an antiderivative of \( \frac{6 + x}{x^2 + 4} \).

Question 2
Find all real solutions of the equation \( 2 \cos(x) = \sqrt{3} \cot(x) \).
Question 3
Consider the equation $z^3 - z^2 - 2z - 12 = 0, z \in C$.

a. Given that $z = 2\text{cis}\left(\frac{2\pi}{3}\right)$ is a root of the equation, find the other two roots in the form $a + ib$, where $a, b \in R$.

b. Plot all of the roots clearly on the Argand diagram below.
Question 4
A crate of mass 50 kg sits on a rough floor. A tension force of $T$ newtons is applied to the crate at $30^\circ$ to the horizontal. The coefficient of friction is 0.2.

a. On the diagram below, show all forces acting on the crate and label them.

b. Write down the maximum value of $T$ that can be applied without the crate leaving the floor.

c. Find the value of $T$ required for the crate to be on the point of moving. Give your answer in the form $\frac{ag}{1 + b\sqrt{c}}$, where $a$, $b$ and $c$ are integers, and $g$ is the acceleration due to gravity.
Question 5
Let \( y = \arctan(2x) \).
Find the value of \( a \) given that \( \frac{d^2 y}{dx^2} = ax \left( \frac{dy}{dx} \right)^2 \), where \( a \) is a real constant.

Question 6
Find the gradient of the tangent to the curve \( xy^2 + y + (\log_x(x - 2))^2 = 14 \) at the point \((3, 2)\).
Question 7
Consider the curve with equation \( y = (x - 1)\sqrt{2-x}, \ 1 \leq x \leq 2. \)
Calculate the area of the region enclosed by the curve and the x-axis.

3 marks

Question 8
The velocity, \( v \) m/s, of a body when it is \( x \) metres from a fixed point \( O \) is given by
\[
v = \frac{2x}{\sqrt{1+x^2}}.
\]
Find an expression for the acceleration of the body in terms of \( x \) in simplest form.

3 marks
Question 9

The position of a particle at time $t$ is given by

$$r(t) = \left(2\sqrt{t^2 + 2} - t^2 \right) \mathbf{i} + \left(2\sqrt{t^2 + 2} + 2t\right) \mathbf{j}, \ t \geq 0.$$ 

a. Find the velocity of the particle at time $t$.

b. Find the speed of the particle at time $t = 1$ in the form $\frac{a\sqrt{b}}{c}$, where $a$, $b$ and $c$ are positive integers.

c. Show that at time $t = 1$, $\frac{dy}{dx} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$.

d. Find the angle in terms of $\pi$, between the vector $-\sqrt{3} \mathbf{i} + \mathbf{j}$ and the vector $r(t)$ at time $t = 0.$
Question 10
Consider the functions with rules \( f(x) = \arcsin \left( \frac{x}{2} \right) + \frac{3}{\sqrt{25x^2 - 1}} \) and \( g(x) = \arcsin(3x) - \frac{3}{\sqrt{25x^2 - 1}} \).

a. i. Find the maximal domain of \( f_1(x) = \arcsin \left( \frac{x}{2} \right) \).

ii. Find the maximal domain of \( f_2(x) = \frac{3}{\sqrt{25x^2 - 1}} \).

iii. Find the largest set of values of \( x \in \mathbb{R} \) for which \( f(x) \) is defined.

b. Given that \( h(x) = f(x) + g(x) \) and that \( \theta = h \left( \frac{1}{4} \right) \), evaluate \( \sin(\theta) \).

Give your answer in the form \( \frac{a\sqrt{b}}{c} \), \( a, b, c \in \mathbb{Z} \).
SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.
Specialist Mathematics Formulas

Mensuration

area of a trapezium: \( \frac{1}{2}(a + b)h \)

curved surface area of a cylinder: \( 2\pi rh \)

volume of a cylinder: \( \pi r^2 h \)

volume of a cone: \( \frac{1}{3}\pi r^2 h \)

volume of a pyramid: \( \frac{1}{3}Ah \)

volume of a sphere: \( \frac{4}{3}\pi r^3 \)

area of a triangle: \( \frac{1}{2}bc \sin A \)

cosec rule: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)

cosine rule: \( c^2 = a^2 + b^2 - 2ab \cos C \)

Coordinate geometry

ellipse: \( \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \)

hyperbola: \( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \)

Circular (trigonometric) functions

\( \cos^2(x) + \sin^2(x) = 1 \)

\( 1 + \tan^2(x) = \sec^2(x) \)

\( \cot^2(x) + 1 = \csc^2(x) \)

\( \sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y) \)

\( \sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y) \)

\( \cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y) \)

\( \cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y) \)

\( \tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)} \)

\( \tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)} \)

\( \cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x) \)

\( \sin(2x) = 2 \sin(x) \cos(x) \)

\( \tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)} \)

<table>
<thead>
<tr>
<th>function</th>
<th>( \sin^{-1} )</th>
<th>( \cos^{-1} )</th>
<th>( \tan^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
<td>([-1, 1])</td>
<td>([-1, 1])</td>
<td>(\mathbb{R})</td>
</tr>
<tr>
<td>range</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
<td>([0, \pi])</td>
<td>((-\frac{\pi}{2}, \frac{\pi}{2}))</td>
</tr>
</tbody>
</table>
Algebra (complex numbers)

\[ z = x + yi = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta \]

\[ |z| = \sqrt{x^2 + y^2} = r \]

\[-\pi < \text{Arg} z \leq \pi \]

\[ z_1z_2 = r_1r_2 \operatorname{cis}(\theta_1 + \theta_2) \]

\[ z^n = r^n \operatorname{cis}(n\theta) \quad (\text{de Moivre’s theorem}) \]

Calculus

\[ \frac{d}{dx} (x^n) = nx^{n-1} \]

\[ \frac{d}{dx} (e^{ax}) = ae^{ax} \]

\[ \frac{d}{dx} (\log_e(x)) = \frac{1}{x} \]

\[ \frac{d}{dx} (\sin(ax)) = a\cos(ax) \]

\[ \frac{d}{dx} (\cos(ax)) = -a\sin(ax) \]

\[ \frac{d}{dx} (\tan(ax)) = a\sec^2(ax) \]

\[ \frac{d}{dx} (\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx} (\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx} (\tan^{-1}(x)) = \frac{1}{1+x^2} \]

\[ \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

\[ \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \]

Euler’s method:

If \( \frac{dy}{dx} = f(x) \), \( y_0 = a \) and \( y_0 = b \), then \( x_{n+1} = x_n + h \) and \( y_{n+1} = y_n + hf(x_n) \)

Acceleration:

\[ a = \frac{dv}{dt} = \frac{dy}{dx} = \frac{d}{dx} \left( \frac{1}{2}v^2 \right) \]

Constant (uniform) acceleration:

\[ v = u + at \quad s = ut + \frac{1}{2}at^2 \quad v^2 = u^2 + 2as \quad s = \frac{1}{2} (u + v)t \]
Vectors in two and three dimensions

\[ \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \]

\[ |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = r \]

\[ \mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos \theta = x_1 x_2 + y_1 y_2 + z_1 z_2 \]

Mechanics

momentum: \[ \mathbf{p} = m\mathbf{v} \]

equation of motion: \[ \mathbf{R} = m\mathbf{a} \]

friction: \[ F \leq \mu N \]