SPECIALIST MATHEMATICS

Written examination 1

Friday 7 November 2014
Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

<table>
<thead>
<tr>
<th>Number of questions</th>
<th>Number of questions to be answered</th>
<th>Number of marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>8</td>
<td>40</td>
</tr>
</tbody>
</table>

• Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
• Students are not permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied
• Question and answer book of 10 pages with a detachable sheet of miscellaneous formulas in the centrefold.
• Working space is provided throughout the book.

Instructions
• Detach the formula sheet from the centre of this book during reading time.
• Write your student number in the space provided above on this page.
• All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Question 1 (5 marks)
Consider the vector \( \mathbf{a} = \sqrt{3} \mathbf{i} - \mathbf{j} - \sqrt{2} \mathbf{k} \), where \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) are unit vectors in the positive directions of the \( x \), \( y \) and \( z \) axes respectively.

a. Find the unit vector in the direction of \( \mathbf{a} \).  
   1 mark

b. Find the acute angle that \( \mathbf{a} \) makes with the positive direction of the \( x \)-axis.  
   2 marks

c. The vector \( \mathbf{b} = 2\sqrt{3} \mathbf{i} + mj - 5\mathbf{k} \).
   Given that \( \mathbf{b} \) is perpendicular to \( \mathbf{a} \), find the value of \( m \).  
   2 marks
Question 2 (5 marks)
The position vector of a particle at time \( t \geq 0 \) is given by
\[
\mathbf{r}(t) = (t - 2) \mathbf{i} + (t^2 - 4t + 1) \mathbf{j}
\]

a. Show that the cartesian equation of the path followed by the particle is \( y = x^2 - 3 \). 
   1 mark

b. Sketch the path followed by the particle on the axes below, labelling all important features. 
   2 marks

\[
\begin{array}{c}
\text{y} \\
\text{O} \\
\text{x}
\end{array}
\]

c. Find the speed of the particle when \( t = 1 \). 
   2 marks
Question 3 (5 marks)
Let \( f \) be a function of a complex variable, defined by the rule \( f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6 \).

a. Given that \( z = i \) is a solution of \( f(z) = 0 \), write down a quadratic factor of \( f(z) \). 2 marks

b. Given that the other quadratic factor of \( f(z) \) has the form \( z^2 + bz + c \), find all solutions of \( z^4 - 4z^3 + 7z^2 - 4z + 6 = 0 \) in cartesian form. 3 marks
**Question 4** (3 marks)
Find the gradient of the normal to the curve defined by \( y = -3e^{3x}e^y \) at the point \((1, -3)\).
Question 5 (5 marks)

a. For the function with rule \( f(x) = 96\cos(3x)\sin(3x) \), find the value of \( a \) such that \( f(x) = a\sin(6x) \).  

b. Use an appropriate substitution in the form \( u = g(x) \) to find an equivalent definite integral for \( \int_{\pi/6}^{\pi} 96\cos(3x)\sin(3x)\cos^2(6x) \, dx \) in terms of \( u \) only.

c. Hence evaluate \( \int_{\pi/6}^{\pi} 96\cos(3x)\sin(3x)\cos^2(6x) \, dx \), giving your answer in the form \( \sqrt{k} \), \( k \in \mathbb{Z} \).
Question 6 (5 marks)

a. Verify that \( \frac{a}{a - 4} = 1 + \frac{4}{a - 4} \).  

b. The region enclosed by the graph of \( y = \frac{x}{\sqrt{x^2 - 4}} \) and the lines \( y = 0, x = 3 \) and \( x = 4 \) is rotated about the x-axis.

Find the volume of the resulting solid of revolution.
Question 7 (5 marks)
Consider \( f(x) = 3x \arctan (2x) \).

a. Write down the range of \( f \). 

b. Show that \( f'(x) = 3 \arctan (2x) + \frac{6x}{1+4x^2} \).

c. **Hence** evaluate the area enclosed by the graph of \( g(x) = \arctan (2x) \), the \( x \)-axis and the lines \( x = \frac{1}{2} \) and \( x = \frac{\sqrt{3}}{2} \).
**Question 8 (7 marks)**

A body of mass 5 kg is held in equilibrium by two light inextensible strings. One string is attached to a ceiling at $A$ and the other to a wall at $B$. The string attached to the ceiling is at an angle $\theta$ to the vertical and has tension $T_1$ newtons, and the other string is horizontal and has tension $T_2$ newtons. Both strings are made of the same material.

![Diagram of a body held in equilibrium by two strings, with one string attached to the ceiling at $A$ and the other to the wall at $B$. The string attached to the ceiling forms an angle $\theta$ with the vertical.]

**a.**

i. Resolve the forces on the body vertically and horizontally, and express $T_1$ in terms of $\theta$.  
2 marks

ii. Express $T_2$ in terms of $\theta$.  
1 mark

**b.**

Show that $\tan(\theta) < \sec(\theta)$ for $0 < \theta < \frac{\pi}{2}$.  
1 mark

**c.**

The type of string used will break if it is subjected to a tension of more than 98 N. Find the maximum allowable value of $\theta$ so that neither string will break.  
3 marks
FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.
Specialist Mathematics formulas

Mensuration

area of a trapezium: \( \frac{1}{2}(a + b)h \)
curved surface area of a cylinder: \( 2\pi rh \)
volume of a cylinder: \( \pi r^2h \)
volume of a cone: \( \frac{1}{3}\pi r^2h \)
volume of a pyramid: \( \frac{1}{3}Ah \)
volume of a sphere: \( \frac{4}{3}\pi r^3 \)
area of a triangle: \( \frac{1}{2}bc \sin A \)
sine rule: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)
cosine rule: \( c^2 = a^2 + b^2 - 2ab \cos C \)

Coordinate geometry

ellipse: \( \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \)
hyperbola: \( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \)

Circular (trigonometric) functions

\( \cos^2(x) + \sin^2(x) = 1 \)

\( 1 + \tan^2(x) = \sec^2(x) \)
\( \cot^2(x) + 1 = \cosec^2(x) \)
\( \sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y) \)
\( \sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y) \)
\( \cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y) \)
\( \cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y) \)
\( \tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)} \)
\( \tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)} \)
\( \cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x) \)
\( \sin(2x) = 2 \sin(x) \cos(x) \)
\( \tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)} \)

<table>
<thead>
<tr>
<th>function</th>
<th>( \sin^{-1} )</th>
<th>( \cos^{-1} )</th>
<th>( \tan^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
<td>([-1, 1])</td>
<td>([-1, 1])</td>
<td>( \mathbb{R} )</td>
</tr>
<tr>
<td>range</td>
<td>([\frac{-\pi}{2}, \frac{\pi}{2}])</td>
<td>([0, \pi])</td>
<td>( \left(\frac{-\pi}{2}, \frac{\pi}{2}\right))</td>
</tr>
</tbody>
</table>
Algebra (complex numbers)

\[ z = x + yi = r(\cos \theta + i \sin \theta) = r \text{cis} \theta \]

\[ |z| = \sqrt{x^2 + y^2} = r \quad -\pi < \text{Arg} \ z \leq \pi \]

\[ z_1z_2 = r_1r_2 \text{cis}(\theta_1 + \theta_2) \]

\[ z^n = r^n \text{cis}(n\theta) \quad (\text{de Moivre’s theorem}) \]

Calculus

\[ \frac{d}{dx}(x^n) = nx^{n-1} \quad \int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1 \]

\[ \frac{d}{dx}(e^{ax}) = ae^{ax} \quad \int e^{ax} \, dx = \frac{1}{a}e^{ax} + c \]

\[ \frac{d}{dx}(\log_c(x)) = \frac{1}{x} \quad \int \frac{1}{x} \, dx = \log_c |x| + c \]

\[ \frac{d}{dx}(\sin(ax)) = a\cos(ax) \quad \int \sin(ax) \, dx = -\frac{1}{a}\cos(ax) + c \]

\[ \frac{d}{dx}(\cos(ax)) = -a\sin(ax) \quad \int \cos(ax) \, dx = \frac{1}{a}\sin(ax) + c \]

\[ \frac{d}{dx}(\tan(ax)) = a\sec^2(ax) \quad \int \sec^2(ax) \, dx = \frac{1}{a}\tan(ax) + c \]

\[ \frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad \int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left( \frac{x}{a} \right) + c, a > 0 \]

\[ \frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}} \quad \int -\frac{1}{\sqrt{a^2-x^2}} \, dx = \cos^{-1}\left( \frac{x}{a} \right) + c, a > 0 \]

\[ \frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} \quad \int \frac{a}{a^2+x^2} \, dx = \tan^{-1}\left( \frac{x}{a} \right) + c \]

\[ \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

\[ \frac{d}{dx}\left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \]

Euler’s method:

If \( \frac{dy}{dx} = f(x) \), \( x_0 = a \) and \( y_0 = b \), then \( x_{n+1} = x_n + h \) and \( y_{n+1} = y_n + hf(x_n) \)

Acceleration:

\[ a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx} \left( \frac{1}{2}v^2 \right) \]

Constant (uniform) acceleration:

\[ v = u + at \quad s = ut + \frac{1}{2}at^2 \quad v^2 = u^2 + 2as \quad s = \frac{1}{2} (u + v)t \]
Vectors in two and three dimensions

\[ \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \]

\[ |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = r \]

\[ \mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos \theta = x_1 x_2 + y_1 y_2 + z_1 z_2 \]

\[ \mathbf{i} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \]

Mechanics

momentum:

\[ \mathbf{p} = m\mathbf{v} \]

equation of motion:

\[ \mathbf{R} = m\mathbf{a} \]

friction:

\[ F \leq \mu N \]