SPECIALIST MATHEMATICS

Written examination 1

Friday 27 October 2006

Reading time: 3.00 pm to 3.15 pm (15 minutes)
Writing time: 3.15 pm to 4.15 pm (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

<table>
<thead>
<tr>
<th>Number of questions</th>
<th>Number of questions to be answered</th>
<th>Number of marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>9</td>
<td>40</td>
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</tbody>
</table>

• Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
• Students are not permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied
• Question and answer book of 10 pages with a detachable sheet of miscellaneous formulas in the centrefold.
• Working space is provided throughout the book.

Instructions
• Detach the formula sheet from the centre of this book during reading time.
• Write your student number in the space provided above on this page.
• All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.
Question 1
Consider the relation $2xy - 9y^2 + 9 = 0$.

a. Find an expression for $\frac{dy}{dx}$ in terms of $x$ and $y$.

b. Hence find the exact value of $\frac{dy}{dx}$ when $y = 1$. 
Question 2

Solve the differential equation \( \frac{dy}{dx} = x\sqrt{x^2 - 16}, \ x \geq 4 \) given that \( y = 13 \) when \( x = 5 \).
Question 3

a. Sketch the graph with equation \( y = \frac{36}{2x^2 - 18} \), clearly indicating the location of any asymptotes and intercepts with the axes.

b. Find the exact area bounded by \( y = \frac{36}{2x^2 - 18} \), the x-axis and the lines \( x = -2 \) and \( x = 2 \).
**Question 4**
A model glider with a mass of 4 kg is suspended from the roof of a wind tunnel by a thin wire. A light wind exerts a horizontal force of magnitude \( \frac{g}{2} \) newtons on the model glider.

a. Let \( T \) newtons be the magnitude of the tension in the suspending wire. Clearly label all forces acting on the model glider on the following diagram.

![Diagram of a model glider with forces labeled](image)

1 mark

b. Calculate the value of \( T \), giving your answer in the form \( \frac{g \sqrt{a}}{b} \) where \( a \) and \( b \) are positive integers.

2 marks
Question 5

a. Show that \( \tan \left( \frac{\pi}{8} \right) = \sqrt{2} - 1 \). 

b. If \( y = \tan^{-1}(x - 1) + a \tan \left( \frac{\pi}{8} \right) \), where \( a \) is a real constant, find the minimum value of \( a \) for which \( y > 0 \) for all \( x \).
Question 6
The region in the first quadrant enclosed by the coordinate axes, the graph with equation \( y = e^{-x} \) and the straight line \( x = a \) where \( a > 0 \), is rotated about the \( x \)-axis to form a solid of revolution.

a. Express the volume of the solid of revolution as a definite integral.

b. Calculate the volume of the solid of revolution, in terms of \( a \).

c. Find the exact value of \( a \) if the volume is \( \frac{5\pi}{18} \) cubic units.
Question 7

The position vector of a moving particle is given by \( \mathbf{r}(t) = \sqrt{t-2} \mathbf{i} + 2t \mathbf{j} \) for \( 2 \leq t \leq 6 \).

a. Find the cartesian equation of the path followed by the particle.

b. Sketch the path of the particle on the axes provided.

\[ 
\begin{align*}
\text{y}& \hspace{1cm} \text{x} \\
\end{align*}
\]

2 marks

2 marks
Question 8
Find an antiderivative of \( \frac{2 + 6x}{\sqrt{4 - x^2}} \).


4 marks

Question 9
a. Express \( 1 + \sqrt{3} \, i \) in polar form.


1 mark

b. Solve the quadratic equation \( z^2 + 2z - \sqrt{3} \, i = 0 \), expressing your answers in exact cartesian form.


3 marks
SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.
Specialist Mathematics Formulas

Mensuration

area of a trapezium: \( \frac{1}{2}(a + b)h \)
curved surface area of a cylinder: \( 2\pi rh \)
volume of a cylinder: \( \pi r^2 h \)
volume of a cone: \( \frac{1}{3}\pi r^2 h \)
volume of a pyramid: \( \frac{1}{3}Ah \)
volume of a sphere: \( \frac{4}{3}\pi r^3 \)
area of a triangle: \( \frac{1}{2}bc \sin A \)
sine rule: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)
cosine rule: \( c^2 = a^2 + b^2 - 2ab \cos C \)

Coordinate geometry

ellipse: \( \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \)  
hyperbola: \( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \)

Circular (trigonometric) functions

\( \cos^2(x) + \sin^2(x) = 1 \)
\( 1 + \tan^2(x) = \sec^2(x) \)
\( \cot^2(x) + 1 = \cosec^2(x) \)
\( \sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y) \)
\( \sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y) \)
\( \cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y) \)
\( \cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y) \)
\( \tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)} \)
\( \tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)} \)
\( \cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x) \)
\( \sin(2x) = 2 \sin(x) \cos(x) \)
\( \tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)} \)

<table>
<thead>
<tr>
<th>function</th>
<th>( \sin^{-1} )</th>
<th>( \cos^{-1} )</th>
<th>( \tan^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
<td>([-1, 1])</td>
<td>([-1, 1])</td>
<td>(\mathbb{R})</td>
</tr>
<tr>
<td>range</td>
<td>(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right])</td>
<td>([0, \pi])</td>
<td>(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right))</td>
</tr>
</tbody>
</table>
Algebra (complex numbers)

\[ z = x + yi = r(\cos \theta + i \sin \theta) = r \, \text{cis} \, \theta \]

\[ |z| = \sqrt{x^2 + y^2} = r \]

\[ -\pi < \text{Arg} \, z \leq \pi \]

\[ z_1 z_2 = r_1 r_2 \, \text{cis}(\theta_1 + \theta_2) \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} \, \text{cis}(\theta_1 - \theta_2) \]

\[ z^n = r^n \, \text{cis}(n\theta) \] (de Moivre’s theorem)

Calculus

\[ \frac{d}{dx} (x^n) = nx^{n-1} \quad \int x^a \, dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1 \]

\[ \frac{d}{dx} (e^{ax}) = ae^{ax} \quad \int e^{ax} \, dx = \frac{1}{a} e^{ax} + c \]

\[ \frac{d}{dx} (\log_a (x)) = \frac{1}{x} \quad \int \frac{1}{x} \, dx = \log_a |x| + c \]

\[ \frac{d}{dx} (\sin(ax)) = a \cos(ax) \quad \int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + c \]

\[ \frac{d}{dx} (\cos(ax)) = -a \sin(ax) \quad \int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + c \]

\[ \frac{d}{dx} (\tan(ax)) = a \sec^2(ax) \quad \int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax) + c \]

\[ \frac{d}{dx} (\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad \int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c, a > 0 \]

\[ \frac{d}{dx} (\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}} \quad \int \frac{-1}{\sqrt{a^2-x^2}} \, dx = \cos^{-1}\left(\frac{x}{a}\right) + c, a > 0 \]

\[ \frac{d}{dx} (\tan^{-1}(x)) = \frac{1}{1+x^2} \quad \int \frac{a}{a^2+x^2} \, dx = \tan^{-1}\left(\frac{x}{a}\right) + c \]

Product rule:

\[ \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

Quotient rule:

\[ \frac{d}{dx} \left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

Chain rule:

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

Euler’s method:

If \( \frac{dy}{dx} = f(x) \), \( x_0 = a \) and \( y_0 = b \), then \( x_{n+1} = x_n + h \) and \( y_{n+1} = y_n + hf(x_n) \)

Acceleration:

\[ a = \frac{d^2x}{dt^2} = \frac{dy}{dt} \cdot \frac{dx}{dt} = \frac{d}{dx} \left( \frac{1}{2} v^2 \right) \]

Constant (uniform) acceleration:

\[ v = u + at \quad s = ut + \frac{1}{2} at^2 \quad v^2 = u^2 + 2as \quad s = \frac{1}{2} (u + v)t \]
Vectors in two and three dimensions

\[ \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \]

\[ |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = r \]

\[ \mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos \theta = x_1 x_2 + y_1 y_2 + z_1 z_2 \]

\[ \mathbf{i} = \frac{dx}{dt} = \frac{d}{dt} x \mathbf{i} + \frac{dy}{dt} = \frac{d}{dt} y \mathbf{j} + \frac{dz}{dt} = \frac{d}{dt} z \mathbf{k} \]

Mechanics

momentum: \[ \mathbf{p} = m\mathbf{v} \]

equation of motion: \[ \mathbf{R} = m\mathbf{a} \]

friction: \[ F \leq \mu \mathbf{N} \]