SPECIALIST MATHEMATICS

Written examination 1

Friday 29 October 2010

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

<table>
<thead>
<tr>
<th>Number of questions</th>
<th>Number of questions to be answered</th>
<th>Number of marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>40</td>
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</tbody>
</table>

• Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
• Students are not permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied
• Question and answer book of 11 pages with a detachable sheet of miscellaneous formulas in the centrefold.
• Working space is provided throughout the book.

Instructions
• Detach the formula sheet from the centre of this book during reading time.
• Write your student number in the space provided above on this page.
• All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.
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Question 1
Consider \( f(z) = z^3 + 9z^2 + 28z + 20, \ z \in \mathbb{C}. \)
Given that \( f(-1) = 0 \), factorize \( f(z) \) over \( \mathbb{C} \).

3 marks
Question 2
A body of mass 2 kg is initially at rest and is acted on by a resultant force of \( v - 4 \) newtons where \( v \) is the velocity in m/s. The body moves in a straight line as a result of the force.

a. Show that the acceleration of the body is given by \( \frac{dv}{dt} = \frac{v - 4}{2} \).

b. Solve the differential equation in part a. to find \( v \) as a function of \( t \).
Question 3
Relative to an origin \( O \), point \( A \) has cartesian coordinates \((1, 2, 2)\) and point \( B \) has cartesian coordinates \((-1, 3, 4)\).

a. Find an expression for the vector \( \overrightarrow{AB} \) in the form \( a \mathbf{i} + b \mathbf{j} + c \mathbf{k} \).

b. Show that the cosine of the angle between the vectors \( \overrightarrow{OA} \) and \( \overrightarrow{AB} \) is \( \frac{4}{9} \).

c. \textbf{Hence} find the exact area of the triangle \( OAB \).
Question 4
Given that $z = 1 + i$, plot and label points for each of the following on the argand diagram below.

i. $z$

ii. $z^2$

iii. $z^4$

Question 5
Given that $f(x) = \arctan(2x)$, find $f^{(n)}\left(\frac{\pi}{2}\right)$.
Question 6

Evaluate \( \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2(2x)\sin(2x) \, dx \).

3 marks
Question 7

Consider the differential equation

\[
\frac{d^2y}{dx^2} = \frac{4x}{(1-x^2)^2}, \quad -1 < x < 1,
\]

for which \( \frac{dy}{dx} = 3 \) when \( x = 0 \), and \( y = 4 \) when \( x = 0 \).

Given that \( \frac{d}{dx} \left( \frac{2}{1-x^2} \right) = \frac{4x}{(1-x^2)^2} \), find the solution of this differential equation.
Question 8
The path of a particle is given by \( \mathbf{r}(t) = t \sin(t) \mathbf{i} - t \cos(t) \mathbf{j}, \ t \geq 0 \). The particle leaves the origin at \( t = 0 \) and then spirals outwards.

a. Show that the second time the particle crosses the \( x \)-axis after leaving the origin occurs when \( t = \frac{3\pi}{2} \).

1 mark

b. Find the speed of the particle when \( t = \frac{3\pi}{2} \).

3 marks

Let \( \theta \) be the acute angle at which the path of the particle crosses the \( x \)-axis.

c. Find \( \tan(\theta) \) when \( t = \frac{3\pi}{2} \).

1 mark
Question 9

a. On the axes below sketch the graph with equation \( x^2 - \frac{(y - 2)^2}{4} = 1 \). State all intercepts with the coordinate axes and give the equations of any asymptotes.

b. Find the gradient of the curve with equation \( x^2 - \frac{(y - 2)^2}{4} = 1 \) at the point where \( x = 2 \) and \( y < 0 \).
Question 10
Part of the graph with equation $y = (x^2 - 1)\sqrt{x + 1}$ is shown below.

Find the area that is bounded by the curve and the $x$-axis. Give your answer in the form $\frac{a\sqrt{b}}{c}$ where $a$, $b$ and $c$ are integers.

\[ \text{4 marks} \]
SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

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Specialist Mathematics Formulas

Mensuration

area of a trapezium: \( \frac{1}{2}(a + b)h \)
curved surface area of a cylinder: \( 2\pi rh \)
volume of a cylinder: \( \pi r^2h \)
volume of a cone: \( \frac{1}{3}\pi r^2h \)
volume of a pyramid: \( \frac{1}{3}Ah \)
volume of a sphere: \( \frac{4}{3}\pi r^3 \)
area of a triangle: \( \frac{1}{2}bc \sin A \)
sine rule: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)
cosine rule: \( c^2 = a^2 + b^2 - 2ab \cos C \)

Coordinate geometry

ellipse: \( \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \)
hyperbola: \( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \)

Circular (trigonometric) functions

\( \cos^2(x) + \sin^2(x) = 1 \)
\( 1 + \tan^2(x) = \sec^2(x) \)
\( \cot^2(x) + 1 = \cosec^2(x) \)
\( \sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y) \)
\( \sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y) \)
\( \cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y) \)
\( \cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y) \)
\( \tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)} \)
\( \tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)} \)
\( \cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x) \)
\( \sin(2x) = 2 \sin(x) \cos(x) \)
\( \tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)} \)

<table>
<thead>
<tr>
<th>function</th>
<th>( \sin^{-1} )</th>
<th>( \cos^{-1} )</th>
<th>( \tan^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
<td>([-1, 1])</td>
<td>([-1, 1])</td>
<td>(\mathbb{R})</td>
</tr>
<tr>
<td>range</td>
<td>([-\frac{\pi}{2}, \frac{\pi}{2}])</td>
<td>([0, \pi])</td>
<td>((-\frac{\pi}{2}, \frac{\pi}{2}))</td>
</tr>
</tbody>
</table>
Algebra (complex numbers)

\[ z = x + yi = r(\cos \theta + i \sin \theta) = r \text{cis} \theta \]

\[ |z| = \sqrt{x^2 + y^2} = r \]

\[ -\pi < \text{Arg} z \leq \pi \]

\[ z_1z_2 = r_1r_2 \text{cis}(\theta_1 + \theta_2) \]

\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2) \]

\[ z^n = r^n \text{cis}(n\theta) \] (de Moivre’s theorem)

Calculus

\[ \frac{d}{dx}(x^n) = nx^{n-1} \]

\[ \frac{d}{dx}(e^{ax}) = ae^{ax} \]

\[ \frac{d}{dx}(\log_c(x)) = \frac{1}{x} \]

\[ \frac{d}{dx}(\sin(ax)) = a \cos(ax) \]

\[ \frac{d}{dx}(\cos(ax)) = -a \sin(ax) \]

\[ \frac{d}{dx}(\tan(ax)) = a \sec^2(ax) \]

\[ \frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} \]

product rule:

\[ \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \]

quotient rule:

\[ \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \]

chain rule:

\[ \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \]

Euler’s method:

If \( \frac{dy}{dx} = f(x) \), \( x_0 = a \) and \( y_0 = b \), then \( x_{n+1} = x_n + h \) and \( y_{n+1} = y_n + hf(x_n) \)

acceleration:

\[ a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx} \left( \frac{1}{2}v^2 \right) \]

constant (uniform) acceleration:

\[ v = u + at \quad s = ut + \frac{1}{2}at^2 \quad v^2 = u^2 + 2as \quad s = \frac{1}{2} (u + v)t \]
Vectors in two and three dimensions

\[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]

\[ |\vec{r}| = \sqrt{x^2 + y^2 + z^2} = r \]

\[ \vec{r}_1 \cdot \vec{r}_2 = |\vec{r}_1| |\vec{r}_2| \cos \theta = x_1x_2 + y_1y_2 + z_1z_2 \]

\[ \vec{r} = \frac{d\vec{x}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \]

Mechanics

momentum:

\[ p = mv \]

equation of motion:

\[ \vec{R} = m\vec{a} \]

friction:

\[ F \leq \mu N \]