VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2012

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Figures										Letter
Words										

MATHEMATICAL METHODS (CAS)

Written examination 2

Thursday 8 November 2012

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 24 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The function with rule $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$ has period

A. 3

B. 5

C. 10

D. $\frac{\pi}{5}$

E. $\frac{\pi}{10}$

Question 2

For the function with rule $f(x) = x^3 - 4x$, the average rate of change of $f(x)$ with respect to x on the interval $[1, 3]$ is

A. 1

B. 3

C. 5

D. 6

E. 9

Question 3

The range of the function $f:[-2, 3) \to R,\ f(x) = x^2 - 2x - 8$ is

A. R

B. $(-9, -5]$

C. $(-5, 0)$

D. $[-9, 0]$

E. $[-9, -5)$

Question 4

Given that g is a differentiable function and k is a real number, the derivative of the composite function $g(e^{kx})$ is

A. $kg'(e^{kx})\, e^{kx}$

B. $kg(e^{kx})$

C. $ke^{kx}\, g(e^{kx})$

D. $ke^{kx}\, g'(e^{x})$

E. $\dfrac{1}{k} e^{kx} g'(e^{kx})$

Question 5

Let the rule for a function g be $g(x) = \log_e((x-2)^2)$. For the function g, the

A. maximal domain = R^+ and range = R

B. maximal domain = $R\backslash\{2\}$ and range = R

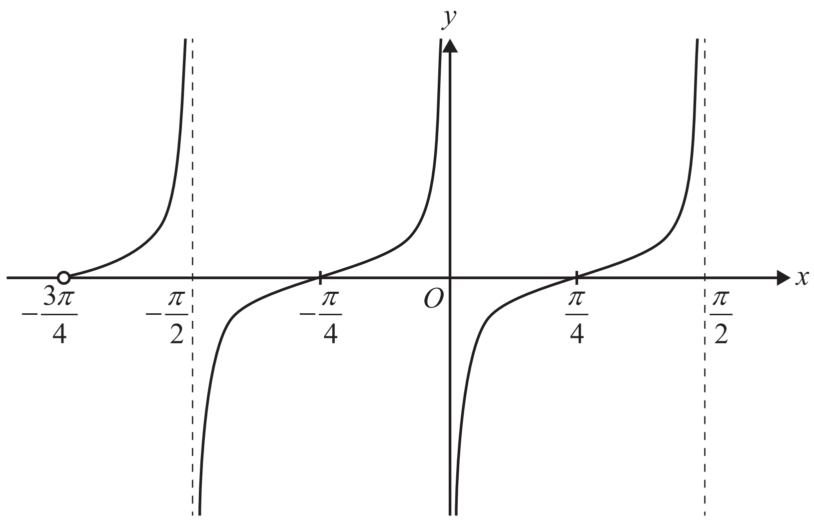
C. maximal domain = $R\backslash\{2\}$ and range = $(-2, \infty)$

D. maximal domain = $[2, \infty)$ and range = $(0, \infty)$

E. maximal domain = $[2, \infty)$ and range = $[0, \infty)$

Question 6

A section of the graph of f is shown below.



The rule of f could be

A. $f(x) = \tan(x)$

B. $f(x) = \tan\left(x - \dfrac{\pi}{4}\right)$

C. $f(x) = \tan\left(2\left(x - \dfrac{\pi}{4}\right)\right)$

D. $f(x) = \tan\left(2\left(x - \dfrac{\pi}{2}\right)\right)$

E. $f(x) = \tan\left(\dfrac{1}{2}\left(x - \dfrac{\pi}{2}\right)\right)$

Question 7

The temperature, T °C, inside a building t hours after midnight is given by the function

$$f:[0, 24] \to R,\ T(t) = 22 - 10\cos\left(\frac{\pi}{12}(t-2)\right)$$

The average temperature inside the building between 2 am and 2 pm is

A. 10 °C
B. 12 °C
C. 20 °C
D. 22 °C
E. 32 °C

Question 8

The function $f: R \to R, f(x) = ax^3 + bx^2 + cx$, where a is a negative real number and b and c are real numbers.
For the real numbers $p < m < 0 < n < q$, we have $f(p) = f(q) = 0$ and $f'(m) = f'(n) = 0$.
The gradient of the graph of $y = f(x)$ is negative for

A. $(-\infty, m) \cup (n, \infty)$
B. (m, n)
C. $(p, 0) \cup (q, \infty)$
D. $(p, m) \cup (0, q)$
E. (p, q)

Question 9

The normal to the graph of $y = \sqrt{b - x^2}$ has a gradient of 3 when $x = 1$.
The value of b is

A. $-\frac{10}{9}$
B. $\frac{10}{9}$
C. 4
D. 10
E. 11

Question 10

The average value of the function $f: [0, 2\pi] \to R, f(x) = \sin^2(x)$ over the interval $[0, a]$ is 0.4.
The value of a, to three decimal places, is

A. 0.850
B. 1.164
C. 1.298
D. 1.339
E. 4.046

Question 11

The weights of bags of flour are normally distributed with mean 252 g and standard deviation 12 g. The manufacturer says that 40% of bags weigh more than x g.

The maximum possible value of x is closest to

A. 249.0

B. 251.5

C. 253.5

D. 254.5

E. 255.0

Question 12

Demelza is a badminton player. If she wins a game, the probability that she will win the next game is 0.7. If she loses a game, the probability that she will lose the next game is 0.6. Demelza has just won a game.

The probability that she will win exactly one of her next two games is

A. 0.33

B. 0.35

C. 0.42

D. 0.49

E. 0.82

Question 13

A and B are events of a sample space S.

$$\Pr(A \cap B) = \frac{2}{5} \text{ and } \Pr(A \cap B') = \frac{3}{7}.$$

$\Pr(B'|A)$ is equal to

A. $\frac{6}{35}$

B. $\frac{15}{29}$

C. $\frac{14}{35}$

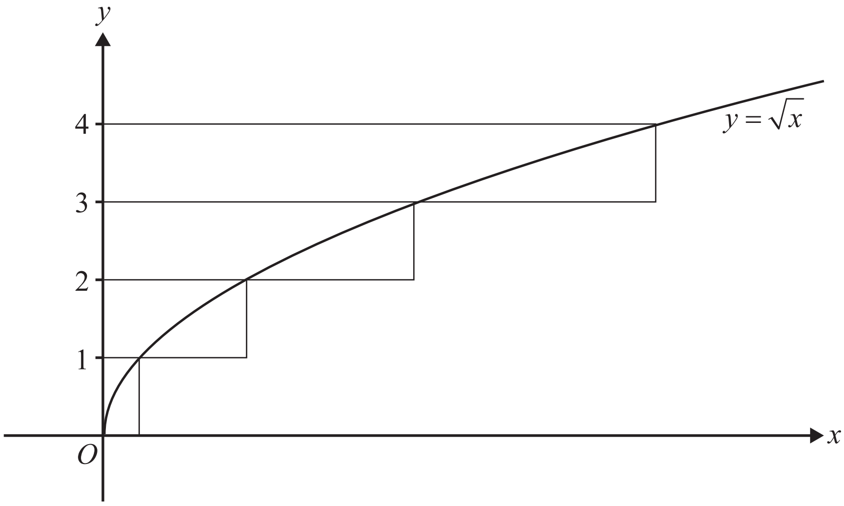
D. $\frac{29}{35}$

E. $\frac{2}{3}$

Question 14

The graph of $f: R^+ \cup \{0\} \to R,\ f(x) = \sqrt{x}$ is shown below.

In order to find an approximation to the area of the region bounded by the graph of f, the y-axis and the line $y = 4$, Zoe draws four rectangles, as shown, and calculates their total area.



Zoe's approximation to the area of the region is

A. 14

B. 21

C. 29

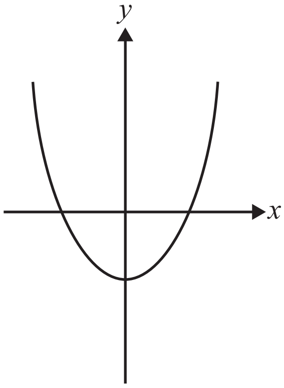
D. 30

E. $\dfrac{64}{3}$

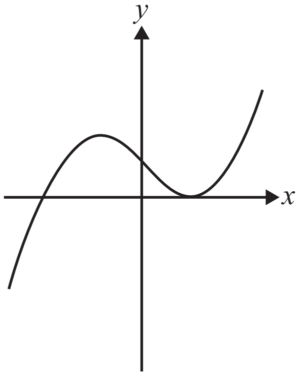
Question 15

If $f'(x) = 3x^2 - 4$, which one of the following graphs could represent the graph of $y = f(x)$?

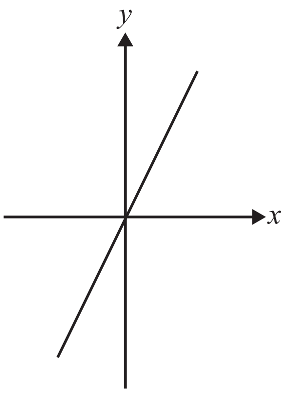
A.



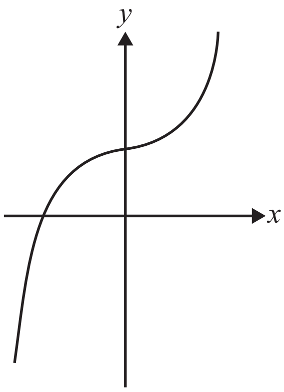
B.



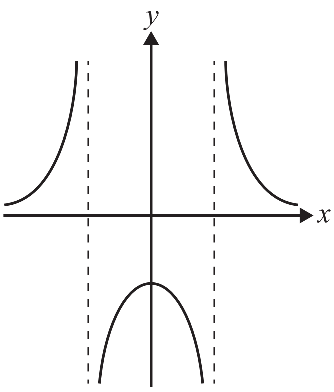
C.



D.



E.



Question 16

The graph of a cubic function f has a local maximum at $(a, -3)$ and a local minimum at $(b, -8)$.

The values of c, such that the equation $f(x) + c = 0$ has exactly one solution, are

A. $3 < c < 8$

B. $c > -3$ or $c < -8$

C. $-8 < c < -3$

D. $c < 3$ or $c > 8$

E. $c < -8$

Question 17

A system of simultaneous linear equations is represented by the matrix equation

$$\begin{bmatrix} m & 3 \\ 1 & m+2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ m \end{bmatrix}.$$

The system of equations will have **no solution** when

A. $m = 1$

B. $m = -3$

C. $m \in \{1, -3\}$

D. $m \in R\backslash\{1\}$

E. $m \in \{1, 3\}$

Question 18

The tangent to the graph of $y = \log_e(x)$ at the point $(a, \log_e(a))$ crosses the x-axis at the point $(b, 0)$, where $b < 0$.
Which of the following is **false**?

A. $1 < a < e$

B. The gradient of the tangent is positive

C. $a > e$

D. The gradient of the tangent is $\dfrac{1}{a}$

E. $a > 0$

Question 19

A function f has the following two properties for all real values of θ.

$$f(\pi - \theta) = -f(\theta) \text{ and } f(\pi - \theta) = -f(-\theta)$$

A possible rule for f is

A. $f(x) = \sin(x)$

B. $f(x) = \cos(x)$

C. $f(x) = \tan(x)$

D. $f(x) = \sin\left(\dfrac{x}{2}\right)$

E. $f(x) = \tan(2x)$

Question 20

A discrete random variable X has the probability function $\Pr(X = k) = (1 - p)^k p$, where k is a non-negative integer.
$\Pr(X > 1)$ is equal to

A. $1 - p + p^2$

B. $1 - p^2$

C. $p - p^2$

D. $2p - p^2$

E. $(1 - p)^2$

Question 21

The volume, V cm^3, of water in a container is given by $V = \frac{1}{3}\pi h^3$ where h cm is the depth of water in the container at time t minutes. Water is draining from the container at a constant rate of 300 cm^3/min. The rate of decrease of h, in cm/min, when $h = 5$ is

A. $\frac{12}{\pi}$

B. $\frac{4}{\pi}$

C. 25π

D. $\frac{60}{\pi}$

E. 30π

Question 22

The graph of a differentiable function f has a local maximum at (a, b), where $a < 0$ and $b > 0$, and a local minimum at (c, d), where $c > 0$ and $d < 0$.

The graph of $y = -|f(x - 2)|$ has

A. a local minimum at $(a - 2, -b)$ and a local maximum at $(c - 2, d)$

B. local minima at $(a + 2, -b)$ and $(c + 2, d)$

C. local maxima at $(a + 2, b)$ and $(c + 2, -d)$

D. a local minimum at $(a - 2, -b)$ and a local maximum at $(a - 2, -d)$

E. local minima at $(c + 2, -d)$ and $(a + 2, -b)$

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

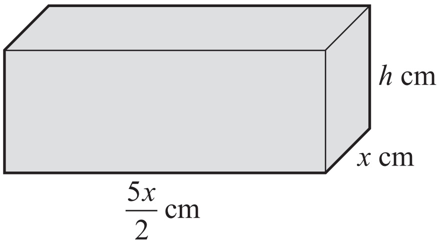
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 sq cm.

a. Show that if the height of the block is h cm, $h = \dfrac{6480 - 5x^2}{7x}$.

2 marks

b. The volume, V cm^3, of the block is given by $V(x) = \dfrac{5x(6480 - 5x^2)}{14}$.

Given that $V(x) > 0$ and $x > 0$, find the possible values of x.

2 marks

c. Find $\dfrac{dV}{dx}$, expressing your answer in the form $\dfrac{dV}{dx} = ax^2 + b$, where a and b are real numbers.

3 marks

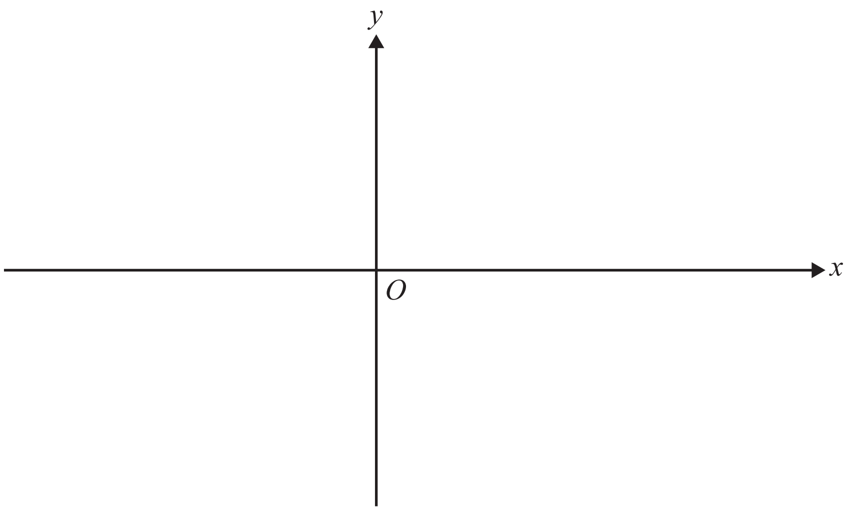
d. Find the exact values of x and h if the block is to have maximum volume.

2 marks

Question 2

Let $f: R\backslash\{2\} \to R,\ f(x) = \dfrac{1}{2x-4} + 3$.

a. Sketch the graph of $y = f(x)$ on the set of axes below. Label the axes intercepts with their coordinates and label each of the asymptotes with its equation.



3 marks

b. **i.** Find $f'(x)$.

ii. State the range of f'.

iii. Using the result of **part ii.** explain why f has no stationary points.

1 + 1 + 1 = 3 marks

c. If (p, q) is any point on the graph of $y = f(x)$, show that the equation of the tangent to $y = f(x)$ at this point can be written as $(2p - 4)^2(y - 3) = -2x + 4p - 4$.

2 marks

d. Find the coordinates of the points on the graph of $y = f(x)$ such that the tangents to the graph at these points intersect at $\left(-1, \frac{7}{2}\right)$.

4 marks

e. A transformation $T: R^2 \to R^2$ that maps the graph of f to the graph of the function $g: R\backslash\{0\} \to R, g(x) = \dfrac{1}{x}$ has rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$, where a, c and d are non-zero real numbers.

Find the values of a, c and d.

2 marks

Question 3

Steve, Katerina and Jess are three students who have agreed to take part in a psychology experiment. Each student is to answer several sets of multiple-choice questions. Each set has the same number of questions, n, where n is a number greater than 20. For each question there are four possible options (A, B, C or D), of which only one is correct.

a. Steve decides to guess the answer to every question, so that for each question he chooses A, B, C or D at random.

Let the random variable X be the number of questions that Steve answers correctly in a particular set.

i. What is the probability that Steve will answer the first three questions of this set correctly?

ii. Find, to four decimal places, the probability that Steve will answer at least 10 of the first 20 questions of this set correctly.

iii. Use the fact that the variance of X is $\frac{75}{16}$ to show that the value of n is 25.

1 + 2 + 1 = 4 marks

If Katerina answers a question correctly, the probability that she will answer the next question correctly is $\frac{3}{4}$. If she answers a question incorrectly, the probability that she will answer the next question incorrectly is $\frac{2}{3}$.

In a particular set, Katerina answers Question 1 incorrectly.

b. **i.** Calculate the probability that Katerina will answer Questions 3, 4 and 5 correctly.

ii. Find the probability that Katerina will answer Question 25 correctly. Give your answer correct to four decimal places.

3 + 2 = 5 marks

c. The probability that Jess will answer any question correctly, independently of her answer to any other question, is p ($p > 0$). Let the random variable Y be the number of questions that Jess answers correctly in any set of 25.

If $\Pr(Y > 23) = 6\Pr(Y = 25)$, show that the value of p is $\frac{5}{6}$.

2 marks

d. From these sets of 25 questions being completed by many students, it has been found that the time, in minutes, that any student takes to answer each set of 25 questions is another random variable, W, which is **normally distributed** with mean a and standard deviation b.

It turns out that, for Jess, $\Pr(Y \geq 18) = \Pr(W \geq 20)$ and also $\Pr(Y \geq 22) = \Pr(W \geq 25)$.

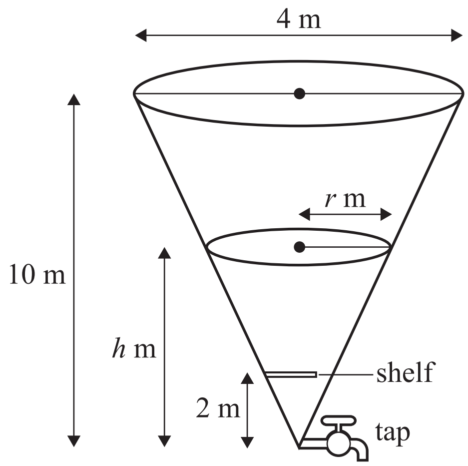
Calculate the values of a and b, correct to three decimal places.

4 marks

Question 4

Tasmania Jones is in the jungle, searching for the Quetzalotl tribe's valuable emerald that has been stolen and hidden by a neighbouring tribe. Tasmania has heard that the emerald has been hidden in a tank shaped like an inverted cone, with a height of 10 metres and a diameter of 4 metres (as shown below).

The emerald is on a shelf. The tank has a poisonous liquid in it.



a. If the depth of the liquid in the tank is h metres

i. find the radius, r metres, of the surface of the liquid in terms of h

ii. show that the volume of the liquid in the tank is $\dfrac{\pi h^3}{75}$ m^3.

1 + 1 = 2 marks

The tank has a tap at its base that allows the liquid to run out of it. The tank is initially full. When the tap is turned on, the liquid flows out of the tank at such a rate that the depth, h metres, of the liquid in the tank is given by

$$h = 10 + \frac{1}{1600}(t^3 - 1200t),$$

where t minutes is the length of time after the tap is turned on until the tank is empty.

b. Show that the tank is empty when $t = 20$.

1 mark

c. When $t = 5$ minutes, find

i. the depth of the liquid in the tank

ii. the rate at which the volume of the liquid is decreasing, correct to one decimal place.

1 + 3 = 4 marks

d. The shelf on which the emerald is placed is 2 metres above the vertex of the cone.

From the moment the liquid starts to flow from the tank, find how long, in minutes, it takes until $h = 2$. (Give your answer correct to one decimal place.)

2 marks

e. As soon as the tank is empty, the tap turns itself off and poisonous liquid starts to flow into the tank at a rate of 0.2 m^3/minute.

How long, in minutes, after the tank is first empty will the liquid once again reach a depth of 2 metres?

2 marks

f. In order to obtain the emerald, Tasmania Jones enters the tank using a vine to climb down the wall of the tank as soon as the depth of the liquid is first 2 metres. He must leave the tank before the depth is again greater than 2 metres.

Find the length of time, in minutes, correct to one decimal place, that Tasmania Jones has from the time he enters the tank to the time he leaves the tank.

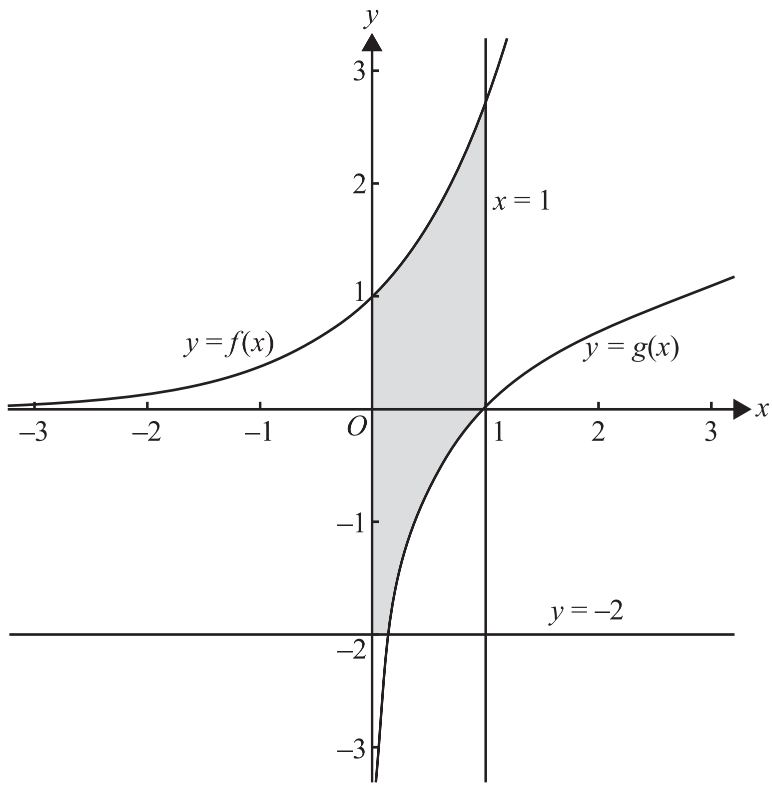
1 mark

Question 5

The shaded region in the diagram below is the plan of a mine site for the Black Possum mining company. All distances are in kilometres.

Two of the boundaries of the mine site are in the shape of the graphs of the functions

$$f: R \to R, f(x) = e^x \text{ and } g: R^+ \to R, g(x) = \log_e(x).$$



a. **i.** Evaluate $\int_{-2}^{0} f(x)\,dx$.

ii. **Hence**, or otherwise, find the area of the region bounded by the graph of g, the x and y axes, and the line $y = -2$.

iii. Find the **total** area of the shaded region.

1 + 1 + 1 = 3 marks

b. The mining engineer, Victoria, decides that a better site for the mine is the region bounded by the graph of g and that of a new function $k: (-\infty, a) \rightarrow R$, $k(x) = -\log_e(a - x)$, where a is a positive real number.

i. Find, in terms of a, the x-coordinates of the points of intersection of the graphs of g and k.

ii. **Hence**, find the set of values of a, for which the graphs of g and k have two distinct points of intersection.

2 + 1 = 3 marks

c. For the new mine site, the graphs of g and k intersect at two distinct points, A and B. It is proposed to start mining operations along the line segment AB, which joins the two points of intersection.

Victoria decides that the graph of k will be such that the x-coordinate of the midpoint of AB is $\sqrt{2}$.

Find the value of a in this case.

2 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

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Mathematical Methods (CAS)
Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

curved surface area of a cylinder: $2\pi rh$

volume of a cylinder: $\pi r^2 h$

volume of a cone: $\frac{1}{3}\pi r^2 h$

volume of a pyramid: $\frac{1}{3}Ah$

volume of a sphere: $\frac{4}{3}\pi r^3$

area of a triangle: $\frac{1}{2}bc\sin A$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\int \frac{1}{x} dx = \log_e|x| + c$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

Probability distribution		**Mean**	**Variance**
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\,p(x)$	$\sigma^2 = \sum (x-\mu)^2\,p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\,f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET