VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2014

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER									Letter

SPECIALIST MATHEMATICS
Written examination 1

Friday 7 November 2014

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
8	8	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are not permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 10 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1 (5 marks)

Consider the vector $\underset{\sim}{a} = \sqrt{3}\,\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\,\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the x, y and z axes respectively.

a. Find the unit vector in the direction of $\underset{\sim}{a}$. 1 mark

b. Find the acute angle that $\underset{\sim}{a}$ makes with the positive direction of the x-axis. 2 marks

c. The vector $\underset{\sim}{b} = 2\sqrt{3}\,\underset{\sim}{i} + m\underset{\sim}{j} - 5\underset{\sim}{k}$.

Given that $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, find the value of m. 2 marks

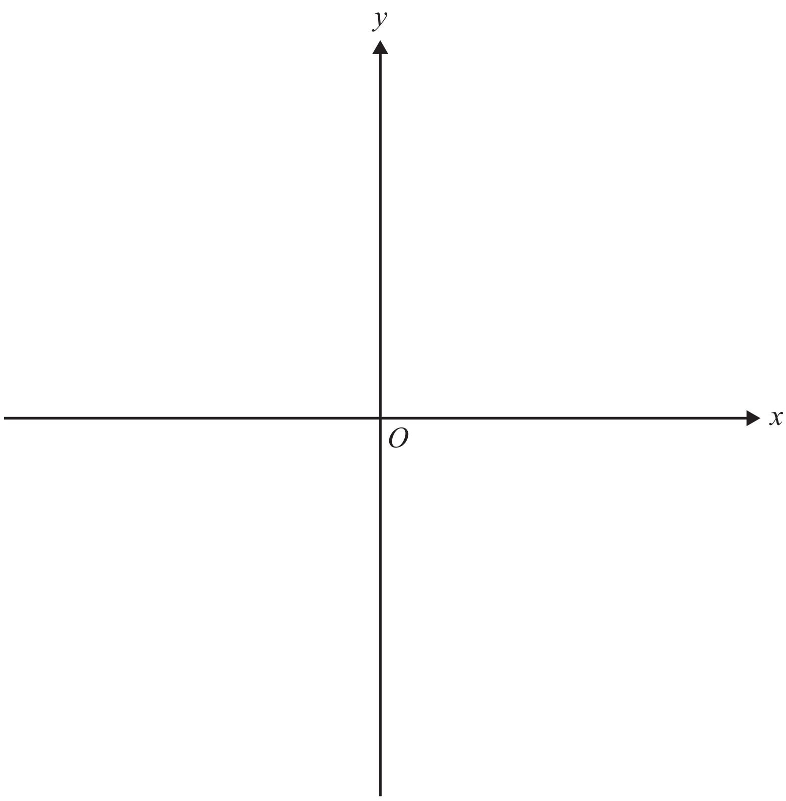
Question 2 (5 marks)

The position vector of a particle at time $t \geq 0$ is given by

$$\underset{\sim}{r}(t) = (t-2)\underset{\sim}{i} + \left(t^2 - 4t + 1\right)\underset{\sim}{j}$$

a. Show that the cartesian equation of the path followed by the particle is $y = x^2 - 3$. 1 mark

b. Sketch the path followed by the particle on the axes below, labelling all important features. 2 marks



c. Find the speed of the particle when $t = 1$. 2 marks

Question 3 (5 marks)

Let f be a function of a complex variable, defined by the rule $f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$.

a. Given that $z = i$ is a solution of $f(z) = 0$, write down a quadratic factor of $f(z)$. 2 marks

b. Given that the other quadratic factor of $f(z)$ has the form $z^2 + bz + c$, find all solutions of $z^4 - 4z^3 + 7z^2 - 4z + 6 = 0$ in cartesian form. 3 marks

Question 4 (3 marks)

Find the gradient of the normal to the curve defined by $y = -3e^{3x}e^{y}$ at the point (1, –3).

Question 5 (5 marks)

a. For the function with rule $f(x) = 96\cos(3x)\sin(3x)$, find the value of a such that $f(x) = a\sin(6x)$.

1 mark

b. Use an appropriate substitution in the form $u = g(x)$ to find an equivalent definite integral for $\int_{\frac{\pi}{36}}^{\frac{\pi}{12}} 96\cos(3x)\sin(3x)\cos^2(6x)\,dx$ in terms of u only.

3 marks

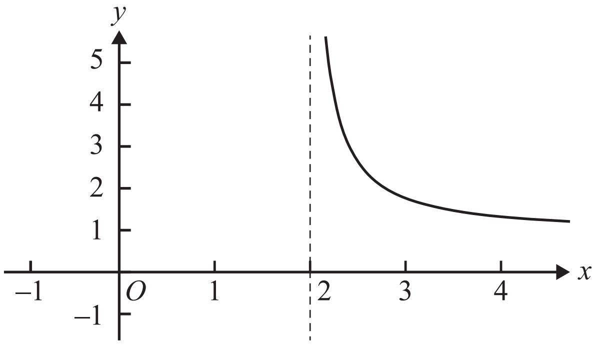
c. **Hence** evaluate $\int_{\frac{\pi}{36}}^{\frac{\pi}{12}} 96\cos(3x)\sin(3x)\cos^2(6x)\,dx$, giving your answer in the form $\sqrt{k}$, $k \in Z$.

1 mark

Question 6 (5 marks)

a. Verify that $\dfrac{a}{a-4}=1+\dfrac{4}{a-4}$. 1 mark

Part of the graph of $y=\dfrac{x}{\sqrt{\left(x^2-4\right)}}$ is shown below.



b. The region enclosed by the graph of $y=\dfrac{x}{\sqrt{\left(x^2-4\right)}}$ and the lines $y=0, x=3$ and $x=4$ is rotated about the x-axis.

Find the volume of the resulting solid of revolution. 4 marks

Question 7 (5 marks)

Consider $f(x) = 3x\arctan(2x)$.

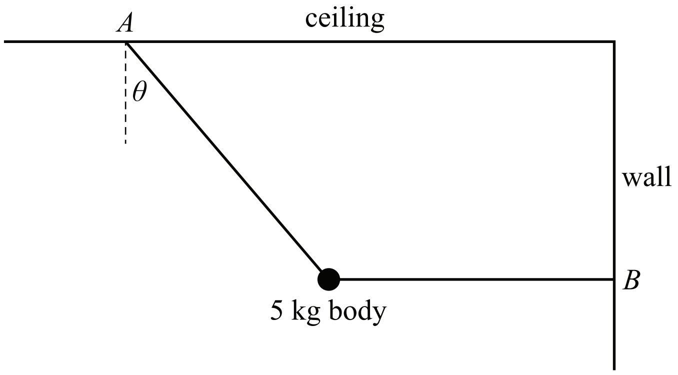
a. Write down the range of f. 1 mark

b. Show that $f'(x) = 3\arctan(2x) + \dfrac{6x}{1+4x^2}$. 1 mark

c. **Hence** evaluate the area enclosed by the graph of $g(x) = \arctan(2x)$, the x-axis and the lines $x = \dfrac{1}{2}$ and $x = \dfrac{\sqrt{3}}{2}$. 3 marks

Question 8 (7 marks)

A body of mass 5 kg is held in equilibrium by two light inextensible strings. One string is attached to a ceiling at A and the other to a wall at B. The string attached to the ceiling is at an angle θ to the vertical and has tension T_1 newtons, and the other string is horizontal and has tension T_2 newtons. Both strings are made of the same material.



a. **i.** Resolve the forces on the body vertically and horizontally, and express T_1 in terms of θ. 2 marks

ii. Express T_2 in terms of θ. 1 mark

b. Show that $\tan(\theta) < \sec(\theta)$ for $0 < \theta < \frac{\pi}{2}$. 1 mark

c. The type of string used will break if it is subjected to a tension of more than 98 N.

Find the maximum allowable value of θ so that **neither** string will break. 3 marks

END OF QUESTION AND ANSWER BOOK

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

Specialist Mathematics formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$
sine rule:	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab\cos C$

Coordinate geometry

ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Circular (trigonometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$ $\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ $\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$ $\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	R
range	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

$|z| = \sqrt{x^2 + y^2} = r$ $\qquad -\pi < \text{Arg}\,z \le \pi$

$z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$ $\qquad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$

$z^n = r^n\,\text{cis}(n\theta)$ (de Moivre's theorem)

Calculus

$\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\qquad \int x^n\,dx = \dfrac{1}{n+1}x^{n+1} + c, n \ne -1$

$\dfrac{d}{dx}\left(e^{ax}\right) = ae^{ax}$ $\qquad \int e^{ax}\,dx = \dfrac{1}{a}e^{ax} + c$

$\dfrac{d}{dx}\left(\log_e(x)\right) = \dfrac{1}{x}$ $\qquad \int \dfrac{1}{x}\,dx = \log_e|x| + c$

$\dfrac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$ $\qquad \int \sin(ax)\,dx = -\dfrac{1}{a}\cos(ax) + c$

$\dfrac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$ $\qquad \int \cos(ax)\,dx = \dfrac{1}{a}\sin(ax) + c$

$\dfrac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$ $\qquad \int \sec^2(ax)\,dx = \dfrac{1}{a}\tan(ax) + c$

$\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad \int \dfrac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c, a > 0$

$\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$ $\qquad \int \dfrac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\dfrac{x}{a}\right) + c, a > 0$

$\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$ $\qquad \int \dfrac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\dfrac{x}{a}\right) + c$

product rule: $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

quotient rule: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Euler's method: If $\dfrac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$

acceleration: $a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

constant (uniform) acceleration: $v = u + at$ $\qquad s = ut + \dfrac{1}{2}at^2$ $\qquad v^2 = u^2 + 2as$ $\qquad s = \dfrac{1}{2}(u + v)t$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$|\underset{\sim}{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$

Mechanics

momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$

equation of motion: $\underset{\sim}{R} = m\underset{\sim}{a}$

friction: $F \le \mu N$

END OF FORMULA SHEET