SPECIALIST MATHEMATICS

Written examination 1

Friday 10 November 2017

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

<table>
<thead>
<tr>
<th>Number of questions</th>
<th>Number of questions to be answered</th>
<th>Number of marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 11 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your student number in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.
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Instructions

Answer all questions in the spaces provided.

Unless otherwise specified, an exact answer is required to a question.

In questions where more than one mark is available, appropriate working must be shown.

Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Take the acceleration due to gravity to have magnitude $g$ ms$^{-2}$, where $g = 9.8$.

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**Question 1** (3 marks)

Find the equation of the tangent to the curve given by $3xy^2 + 2y = x$ at the point $(1, -1)$.

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**Question 2** (4 marks)

Find $\int_{1}^{\sqrt{3}} \frac{1}{x(1+x^2)} dx$, expressing your answer in the form $\log_e \left( \frac{a}{\sqrt{b}} \right)$, where $a$ and $b$ are positive integers.

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TURN OVER
Question 3 (3 marks)
Let \( z^3 + az^2 + 6z + a = 0, \ z \in C, \) where \( a \) is a real constant.
Given that \( z = 1 - i \) is a solution to the equation, find all other solutions.

Question 4 (3 marks)
The volume of soft drink dispensed by a machine into bottles varies normally with a mean of 298 mL and a standard deviation of 3 mL. The soft drink is sold in packs of four bottles.
Find the approximate probability that the mean volume of soft drink per bottle in a randomly selected four-bottle pack is less than 295 mL. Give your answer correct to three decimal places.
Question 5 (4 marks)
Relative to a fixed origin, the points $B$, $C$ and $D$ are defined respectively by the position vectors $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{d} = a\mathbf{i} - 2\mathbf{j}$, where $a$ is a real constant.

Given that the magnitude of angle $BCD$ is $\frac{\pi}{3}$, find $a$. 
Question 6 (3 marks)

Let \( f(x) = \frac{1}{\arcsin(x)} \).

Find \( f'(x) \) and state the largest set of values of \( x \) for which \( f'(x) \) is defined.


Question 7 (4 marks)

The position vector of a particle moving along a curve at time \( t \) is given by \( \mathbf{r}(t) = \cos^3(t)\mathbf{j} + \sin^3(t)\mathbf{j} \), \( 0 \leq t \leq \frac{\pi}{4} \).

Find the length of the path that the particle travels along the curve from \( t = 0 \) to \( t = \frac{\pi}{4} \).
Question 8 (4 marks)

A slope field representing the differential equation \( \frac{dy}{dx} = \frac{-x}{1 + y^2} \) is shown below.

a. Sketch the solution curve of the differential equation corresponding to the condition \( y(-1) = 1 \) on the slope field above and, hence, estimate the positive value of \( x \) when \( y = 0 \). Give your answer correct to one decimal place. 2 marks
b. Solve the differential equation \( \frac{dy}{dx} = \frac{-x}{1+y^2} \) with the condition \( y(-1) = 1 \). Express your answer in the form \( ay^3 + by + cx^2 + d = 0 \), where \( a, b, c \) and \( d \) are integers. 2 marks
Question 9 (5 marks)
A particle of mass 2 kg with initial velocity \(3\mathbf{i} + 2\mathbf{j}\) ms\(^{-1}\) experiences a constant force for 10 seconds.
The particle’s velocity at the end of the 10-second period is \(43\mathbf{i} - 18\mathbf{j}\) ms\(^{-1}\).

a. Find the magnitude of the constant force in newtons. 2 marks

b. Find the displacement of the particle from its initial position after 10 seconds. 3 marks
Question 10 (7 marks)

a. Show that \[ \frac{d}{dx} \left( x \arccos \left( \frac{x}{a} \right) \right) = \arccos \left( \frac{x}{a} \right) - \frac{x}{\sqrt{a^2 - x^2}} \], where \( a > 0 \).

1 mark

b. State the maximal domain and the range of \( f(x) = \sqrt{\arccos \left( \frac{x}{2} \right)} \).

2 marks

c. Find the volume of the solid of revolution generated when the region bounded by the graph of \( y = f(x) \), and the lines \( x = -2 \) and \( y = 0 \), is rotated about the x-axis.

4 marks
SPECIALIST MATHEMATICS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.
Specialist Mathematics formulas

**Mensuration**

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>area of a trapezium</td>
<td>$\frac{1}{2} (a + b) h$</td>
</tr>
<tr>
<td>curved surface area of a cylinder</td>
<td>$2\pi rh$</td>
</tr>
<tr>
<td>volume of a cylinder</td>
<td>$\pi r^2 h$</td>
</tr>
<tr>
<td>volume of a cone</td>
<td>$\frac{1}{3} \pi r^2 h$</td>
</tr>
<tr>
<td>volume of a pyramid</td>
<td>$\frac{1}{3} Ah$</td>
</tr>
<tr>
<td>volume of a sphere</td>
<td>$\frac{4}{3} \pi r^3$</td>
</tr>
<tr>
<td>area of a triangle</td>
<td>$\frac{1}{2} bc \sin(A)$</td>
</tr>
<tr>
<td>sine rule</td>
<td>$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$</td>
</tr>
<tr>
<td>cosine rule</td>
<td>$c^2 = a^2 + b^2 - 2ab \cos(C)$</td>
</tr>
</tbody>
</table>

**Circular functions**

<table>
<thead>
<tr>
<th>Identity</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos^2(x) + \sin^2(x) = 1$</td>
<td></td>
</tr>
<tr>
<td>$1 + \tan^2(x) = \sec^2(x)$</td>
<td>$\cot^2(x) + 1 = \cosec^2(x)$</td>
</tr>
<tr>
<td>$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$</td>
<td>$\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)$</td>
</tr>
<tr>
<td>$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$</td>
<td>$\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$</td>
</tr>
<tr>
<td>$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)}$</td>
<td>$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)}$</td>
</tr>
<tr>
<td>$\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x)$</td>
<td></td>
</tr>
<tr>
<td>$\sin(2x) = 2 \sin(x) \cos(x)$</td>
<td>$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$</td>
</tr>
</tbody>
</table>
Circular functions – continued

<table>
<thead>
<tr>
<th>Function</th>
<th>sin$^{-1}$ or arcsin</th>
<th>cos$^{-1}$ or arccos</th>
<th>tan$^{-1}$ or arctan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>$[-1, 1]$</td>
<td>$[-1, 1]$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>Range</td>
<td>$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$</td>
<td>$[0, \pi]$</td>
<td>$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$</td>
</tr>
</tbody>
</table>

Algebra (complex numbers)

\[ z = x + iy = r \cos(\theta) + i \sin(\theta) = r \text{cis}(\theta) \]

\[ |z| = \sqrt{x^2 + y^2} = r \quad -\pi < \text{Arg}(z) \leq \pi \]

\[ z_1z_2 = r_1r_2 \text{cis}(\theta_1 + \theta_2) \]

\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2) \]

\[ z^n = r^n \text{cis}(n\theta) \quad \text{(de Moivre’s theorem)} \]

Probability and statistics

for random variables $X$ and $Y$

\[ \text{E}(aX + b) = a\text{E}(X) + b \]
\[ \text{E}(aX + bY) = a\text{E}(X) + b\text{E}(Y) \]
\[ \text{var}(aX + b) = a^2\text{var}(X) \]

for independent random variables $X$ and $Y$

\[ \text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y) \]

approximate confidence interval for $\mu$

\[ \left( \bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}} \right) \]

distribution of sample mean $\bar{X}$

mean $\text{E}(\bar{X}) = \mu$

variance $\text{var}(\bar{X}) = \frac{\sigma^2}{n}$
Calculus

\[
\frac{d}{dx}(x^n) = nx^{n-1} \quad \int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, \quad n \neq -1
\]

\[
\frac{d}{dx}(e^{ax}) = ae^{ax} \quad \int e^{ax} \, dx = \frac{1}{a} e^{ax} + c
\]

\[
\frac{d}{dx}(\log_e(x)) = \frac{1}{x} \quad \int \frac{1}{x} \, dx = \log_e |x| + c
\]

\[
\frac{d}{dx}(\sin(ax)) = a \cos(ax) \quad \int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + c
\]

\[
\frac{d}{dx}(\cos(ax)) = -a \sin(ax) \quad \int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + c
\]

\[
\frac{d}{dx}(\tan(ax)) = a \sec^2(ax) \quad \int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax) + c
\]

\[
\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad \int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c, \quad a > 0
\]

\[
\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{a^2-x^2}} \quad \int \frac{-1}{\sqrt{a^2-x^2}} \, dx = \cos^{-1}\left(\frac{x}{a}\right) + c, \quad a > 0
\]

\[
\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} \quad \int \frac{a}{a^2+x^2} \, dx = \tan^{-1}\left(\frac{x}{a}\right) + c
\]

\[
(ax + b)^n \, dx = \frac{1}{a(n+1)} (ax + b)^{n+1} + c, \quad n \neq -1
\]

\[
(ax + b)^{-1} \, dx = \frac{1}{a} \log_e |ax + b| + c
\]

product rule

\[
\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}
\]

quotient rule

\[
\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]

chain rule

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]

Euler’s method

If \( \frac{dy}{dx} = f(x) \), \( x_0 = a \) and \( y_0 = b \), then \( x_{n+1} = x_n + h \) and \( y_{n+1} = y_n + hf(x_n) \)

acceleration

\[
a = \frac{d^2x}{dt^2} = \frac{dv}{dx} = v \frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)
\]

arc length

\[
\int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2} \, dx \quad \text{or} \quad \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt
\]

Vectors in two and three dimensions

\[
\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}
\]

\[
|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = r
\]

\[
\dot{\mathbf{r}} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}
\]

\[
\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1r_2 \cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2
\]

Mechanics

| momentum | \( p = mv \) |
| equation of motion | \( \mathbf{R} = ma \) |

END OF FORMULA SHEET