VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2018

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Letter

STUDENT NUMBER

MATHEMATICAL METHODS

Written examination 2

Monday 4 June 2018

Reading time: 2.00 pm to 2.15 pm (15 minutes)
Writing time: 2.15 pm to 4.15 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	4	4	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 21 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f : R \to R, f(x) = 3 - 2\cos\left(\frac{\pi x}{4}\right)$.

The period and range of this function are respectively

A. 4 and $[-2, 2]$

B. 8 and $[1, 5]$

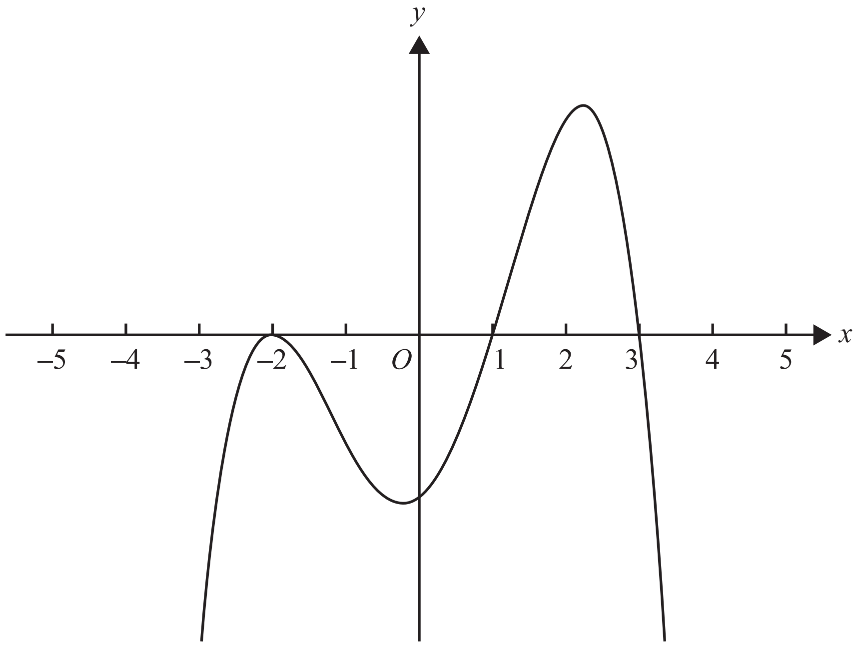
C. 8π and $[1, 5]$

D. 8π and $[-2, 2]$

E. $\frac{1}{2}$ and $[-1, 5]$

Question 2

The diagram below shows part of the graph of a polynomial function.



A possible rule for this function is

A. $y = (x+2)(x-1)(x-3)$

B. $y = (x+2)^2(x-1)(x-3)$

C. $y = (x+2)^2(x-1)(3-x)$

D. $y = -(x-2)^2(x-1)(3-x)$

E. $y = -(x+2)(x-1)(x-3)$

Question 3

A discrete random variable has a binomial distribution with a mean of 3.6 and a variance of 1.98

The values of n (the number of independent trials) and p (the probability of success in each trial) are

A. $n = 4$ and $p = 0.9$

B. $n = 5$ and $p = 0.72$

C. $n = 6$ and $p = 0.6$

D. $n = 8$ and $p = 0.45$

E. $n = 12$ and $p = 0.3$

Question 4

If A and B are events from a sample space such that $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cup B) = 0.7$, then $\Pr(A \cap B')$ is equal to

A. 0.12

B. 0.18

C. 0.2

D. 0.3

E. 0.4

Question 5

A set of three numbers that could be the solutions of $x^3 + ax^2 + 16x + 84 = 0$ is

A. $\{3, 4, 7\}$

B. $\{-4, -3, 7\}$

C. $\{-2, -1, 21\}$

D. $\{-2, 6, 7\}$

E. $\{2, 6, 7\}$

Question 6

The sum of the solutions to the equation $\sqrt{3}\sin(2x) = -3\cos(2x)$ for $x \in [0, 2\pi]$ is equal to

A. $\frac{\pi}{3}$

B. $\frac{7\pi}{6}$

C. $\frac{11\pi}{3}$

D. $\frac{13\pi}{3}$

E. $\frac{14\pi}{3}$

Question 7

Six balls numbered from 1 to 6 are placed in a jar. A ball is taken randomly from the jar and its number is recorded. This ball is returned to the jar, and a second ball is then taken randomly and its number is recorded. The sum of the two recorded numbers is then calculated.

The probability that the sum of the two recorded numbers is 7, given that the first recorded number is odd, is equal to

A. $\frac{1}{3}$

B. $\frac{1}{4}$

C. $\frac{1}{6}$

D. $\frac{1}{12}$

E. $\frac{1}{9}$

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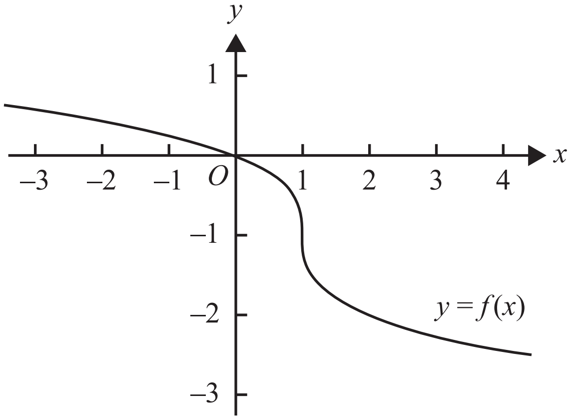
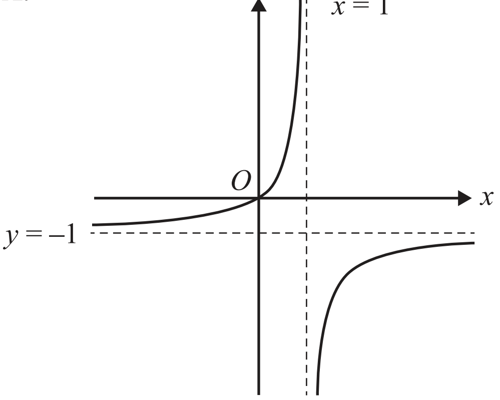
Question 8

Part of the graph of $y = f(x)$ is shown below.

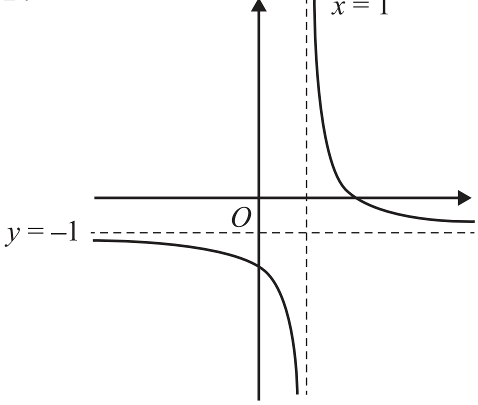


The graph of $y = f'(x)$ is best represented by

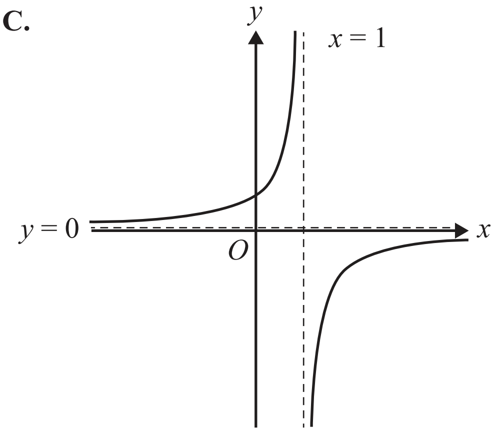
A.



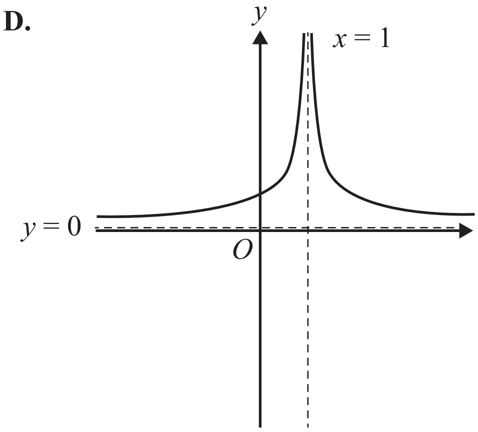
B.



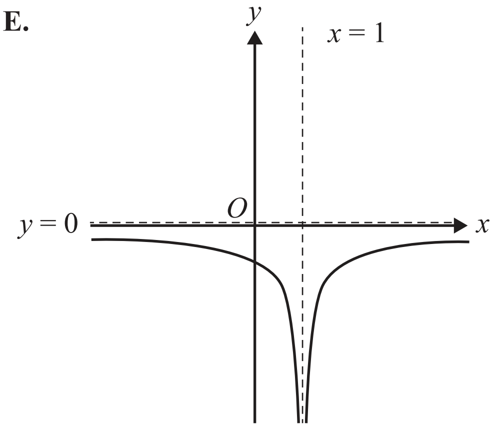
C.



D.



E.



Question 9

A continuous random variable X has a normal distribution with a mean of 40 and a standard deviation of 5. The continuous random variable Z has the standard normal distribution.

$\Pr(-2 < Z < 1)$ is equal to

A. $\Pr(40 < X < 55)$

B. $\Pr(35 < X < 50)$

C. $\Pr(30 < X < 50)$

D. $\Pr(10 < X < 30)$

E. $\Pr(X > 30) - \Pr(X < 45)$

Question 10

The range of the function $f : \left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right] \to R, f(x) = 2x^3 - 3x + 4$ is

A. $\left(4-\sqrt{2}, 4+\sqrt{2}\right)$

B. $\left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right)$

C. $\left(4-\sqrt{2}, 4+\sqrt{2}\right]$

D. $\left(\frac{-1}{\sqrt{2}}, \sqrt{2}\right]$

E. $\left[4-\sqrt{2}, 4+\sqrt{2}\right]$

Question 11

The maximal domain of the function g, where $g(x) = \log_e(-2x)$, is

A. R

B. R^-

C. R^+

D. $[0, \infty)$

E. $(-\infty, 0]$

Question 12

The average value of $f(x) = x^2 - 2x$ over the interval $[1, a]$ is $\frac{13}{3}$.

The value of a is

A. 2

B. 3

C. $\frac{10}{3}$

D. 5

E. $\frac{16}{3}$

Question 13

The function f has the property $f(2x) = (f(x))^2 - 2$ for all real numbers x.

A possible rule for the function $f(x)$ is

A. $\dfrac{1}{x^2+4}$

B. $\cos(x)$

C. $2\log_e(x^2 + 1)$

D. $e^x + e^{-x}$

E. x^2

Question 14

The graph of the function f is obtained from the graph of the function g with rule $g(x) = 3\cos\left(x - \dfrac{\pi}{6}\right)$ by a dilation of a factor of $\dfrac{1}{2}$ from the x-axis, a reflection in the y-axis, a translation of $\dfrac{\pi}{6}$ units in the negative x direction and a translation of 4 units in the negative y direction, in that order.

The rule of f is

A. $f(x) = \dfrac{3}{2}\cos\left(-x - \dfrac{\pi}{3}\right) - 4$

B. $f(x) = \dfrac{3}{2}\cos(-x) - 4$

C. $f(x) = -\dfrac{3}{2}\cos(x) - 4$

D. $f(x) = -3\cos\left(\dfrac{x}{2} - \dfrac{\pi}{3}\right) - 4$

E. $f(x) = \dfrac{3}{2}\cos\left(-x + \dfrac{\pi}{3}\right) - 4$

Question 15

If $\int_{-3}^{2} f(x)\,dx = -8$ and $\int_{2}^{3} f(x)\,dx = 10$, the value of $\int_{-3}^{3} f(x)\,dx$ is

A. 2

B. –2

C. –18

D. 18

E. 0

Question 16

Let $f: R^+ \to R, f(x) = -\log_e(x)$ and $g: R \to R, g(x) = x^2 + 1$.

The domain and range of $f(g(x))$ are respectively

A. R and $R^+ \cup \{0\}$

B. R and R^-

C. $[1, \infty)$ and $R^+ \cup \{0\}$

D. R^+ and $R^+ \cup \{0\}$

E. R and $R^- \cup \{0\}$

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Question 17

If $F(x)$ is an antiderivative of $f(x)$ and $F(4) = -6$, then $F(8)$ is equal to

A. $f'(8) + 6$

B. $-6 + f'(4)$

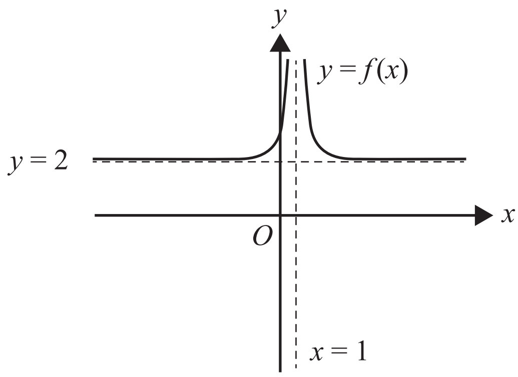
C. $\int_4^8 f(x)dx$

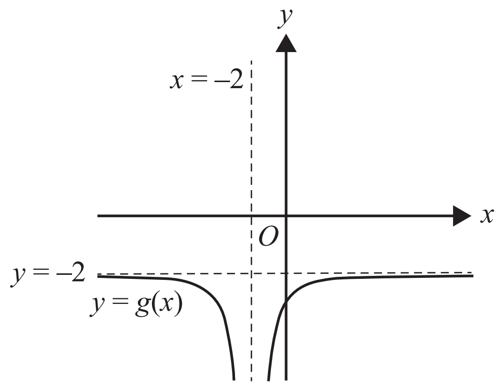
D. $\int_4^8 \left(-6 + f(x)\right)dx$

E. $-6 + \int_4^8 f(x)dx$

Question 18

Consider the graphs of f and g below, which have the same scale.





If T transforms the graph of f onto the graph of g, then

A. $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

B. $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

C. $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

D. $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

E. $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Question 19

A box contains 20 000 marbles that are either blue or red. There are more blue marbles than red marbles. Random samples of 100 marbles are taken from the box. Each random sample is obtained by sampling with replacement.

If the standard deviation of the sampling distribution for the proportion of blue marbles is 0.03, then the number of blue marbles in the box is

A. 11 000

B. 16 000

C. 17 000

D. 18 000

E. 19 000

Question 20

Let f be a one-to-one differentiable function such that $f(3) = 7$, $f(7) = 8$, $f'(3) = 2$ and $f'(7) = 3$.

The function g is differentiable and $g(x) = f^{-1}(x)$ for all x.

$g'(7)$ is equal to

A. $\frac{1}{2}$

B. 2

C. $\frac{1}{6}$

D. $\frac{1}{8}$

E. $\frac{1}{3}$

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

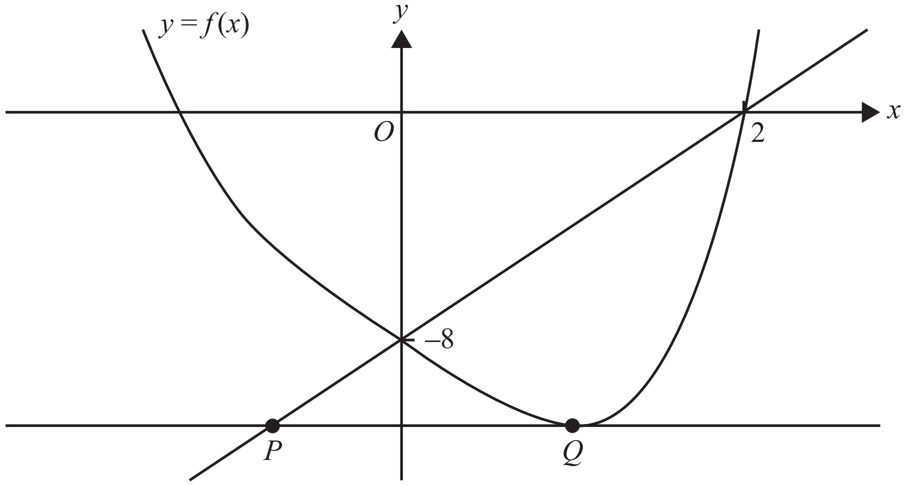
Question 1 (9 marks)

Let $f: R \to R$, $f(x) = x^4 - 4x - 8$.

a. Given $f(x) = (x - 2)(x^3 + ax^2 + bx + c)$, find a, b and c. 1 mark

b. Find two consecutive integers m and n such that a solution to $f(x) = 0$ is in the interval (m, n), where $m < n < 0$. 2 marks

The diagram below shows part of the graph of f and a straight line drawn through the points (0, –8) and (2, 0). A second straight line is drawn parallel to the horizontal axis and it touches the graph of f at the point Q. The two straight lines intersect at the point P.



c. **i.** Find the equation of the line through (0, –8) and (2, 0). 1 mark

ii. State the equation of the line through the points P and Q. 1 mark

iii. State the coordinates of the points P and Q. 2 marks

d. A transformation $T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d \\ 0 \end{bmatrix}$ is applied to the graph of f.

i. Find the value of d for which P is the image of Q. 1 mark

ii. Let $(m', 0)$ and $(n', 0)$ be the images of $(m, 0)$ and $(n, 0)$ respectively, under the transformation T, where m and n are defined in **part b.**

Find the values of m' and n'. 1 mark

Question 2 (18 marks)

Rebecca's Robotics manufactures three types of components for robots: sensors, motors and controllers. The manufacturing processes for each type of component are independent.

It is known that 8% of all of the sensors manufactured are defective.

a. A random sample of five sensors is selected.

Find, correct to four decimal places, the probability that

i. exactly two of these selected sensors are defective 2 marks

ii. exactly two of these selected sensors are defective, given that at most two sensors in the sample are defective. 2 marks

b. A random sample of 50 sensors is selected and it is found that the proportion of defective sensors in this sample is 0.08

Determine an approximate 90% confidence interval for the proportion of defective sensors, correct to four decimal places. 2 marks

A hole is drilled into each motor. The depth of the hole is normally distributed with a mean of 20 mm and a standard deviation of 0.3 mm.

c. What is the probability that, for a randomly selected motor, the depth of the hole is greater than 20.6 mm? Give your answer correct to four decimal places. 1 mark

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The depth of the hole drilled into a motor must be within 0.5 mm of the mean, otherwise the motor is defective.

d. What is the probability that a motor is defective, correct to four decimal places? 2 marks

e. Rebecca delivers an order for five sensors and five motors.

What is the probability that the order contains exactly two defective components? Give your answer correct to three decimal places. 3 marks

f. A knob is attached to each controller. The height of a knob is normally distributed with a mean of 30 mm. If the knob on a controller has a height greater than 30.4 mm or less than 29.6 mm, then the controller is defective.

Rebecca wants to ensure that less than 2% of all controllers manufactured are defective.

What is the maximum standard deviation of the height of a knob, in millimetres, that can be attached to a controller so that less than 2% of controllers are defective? Give your answer correct to two decimal places. 2 marks

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The weight, w, in grams, of controllers is modelled by the following probability density function.

$$C(w)=\begin{cases}\dfrac{3}{640000}(330-w)^2(w-290) & 290\le w\le 330\\ 0 & \text{elsewhere}\end{cases}$$

g. Determine the mean weight, in grams, of the controllers. 2 marks

h. Determine the probability that a randomly selected controller weighs less than the mean weight of the controllers. Give your answer correct to four decimal places. 2 marks

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Question 3 (13 marks)

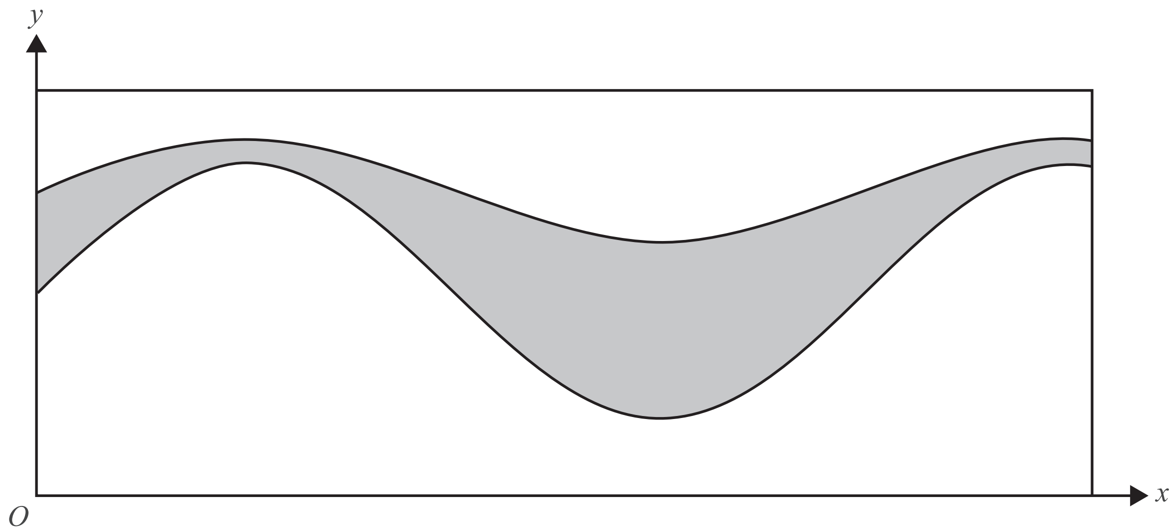
The front of a building has a length of 80 m and a height of 20 m. On the front of the building is a glass panel that lies between two boundary curves, as shown by the shaded region in the diagram below.

The boundary curves of the region are defined over the interval [0, 80] with the rules

$$y_1 = \frac{5}{2}\sin\left(\frac{x}{10}\right) + 15$$

$$y_2 = \frac{25}{4}\sin\left(\frac{x}{10}\right) + 10$$

where x is the horizontal distance, in metres, and y is the vertical distance, in metres, measured relative to an origin, O, at the bottom left corner of the front of the building.



a. Find the total area of the glass panel, in square metres, correct to two decimal places. 2 marks

Let D be the vertical distance between the upper and lower boundary curves.

b. Find the minimum value of D, in metres, and the value(s) of x where this minimum occurs. 3 marks

c. What is the average value of D, in metres, correct to two decimal places? 2 marks

The boundary curves over the interval [0, 80] are generalised to

$$c_1(x) = a\sin\left(\frac{x}{10}\right) + 15$$

$$c_2(x) = a^2\sin\left(\frac{x}{10}\right) + 10$$

where $a \in R^+$.

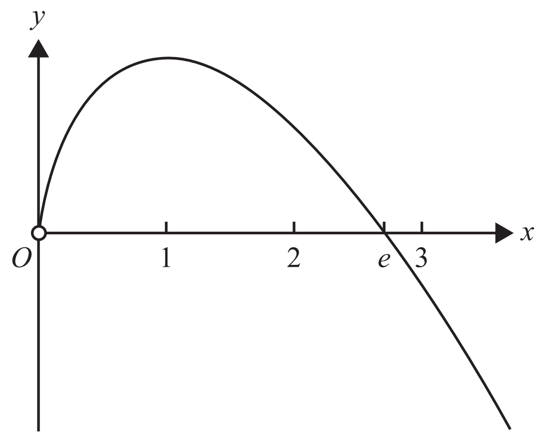
d. The boundary curves do **not** intersect for $a \in (0, p)$.

Find the maximal value of p. 3 marks

e. Find the value of a for which the area of the glass panel is a maximum. Also state the maximum area, in square metres, correct to two decimal places. 3 marks

Question 4 (20 marks)

Let $f:(0, \infty) \to R$, $f(x) = x - x\log_e(x)$. Part of the graph of f is shown below.



a. Find the values of x for which

i. $-1 < f'(x) < -\frac{1}{2}$ 2 marks

ii. $\frac{1}{2} < f'(x) < 1$ 1 mark

b. **i.** Find the equation of the tangent to the graph of f at the point $(a, f(a))$ in the form $y = mx + c$. 1 mark

ii. Find the coordinates of the point of intersection of the tangent to the graph of f at $x = a$ and the tangent to the graph of f at $x = \frac{1}{a}$. 2 marks

iii. Hence, find the coordinates of the point of intersection of the tangents to the graph of f at $x = e$ and $x = \frac{1}{e}$. Express each coordinate in terms of e. 1 mark

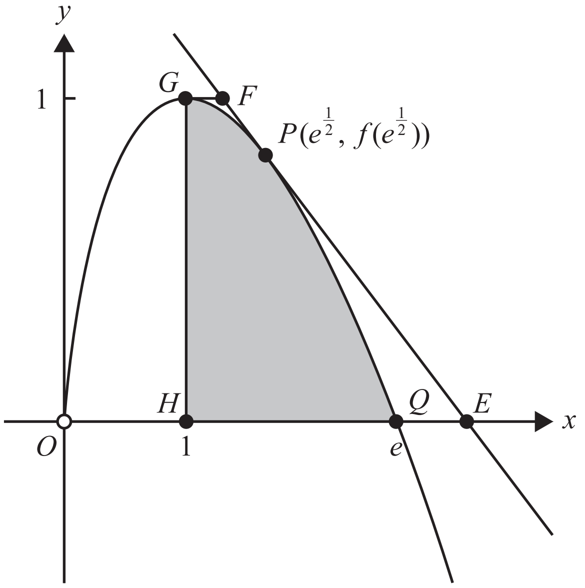
c. **i.** For a value of $b > e$, the tangent to f at the point $(b, f(b))$ and the tangent to f at the point $(2, f(2))$ intersect the x-axis at the same point.

Find the value of b. 2 marks

ii. If the tangent to f at the point $(p, f(p))$, where $1 < p < e$, and the tangent to f at the point $(q, f(q))$, where $q > e$, intersect on the x-axis, show that $p^q = q^p$. 2 marks

d. Find the equation of the tangent to the graph of f at the point where $x = e^{\frac{1}{2}}$. 1 mark

e. Part of the graph of f, with the tangent to the graph at P where $x = e^{\frac{1}{2}}$, is shown below.
E is the point corresponding to the x-axis intercept of this tangent.
F is the point on this tangent where $y = 1$.
G is the point corresponding to the local maximum of the graph of f.
H is the point (1, 0).
Q is the point (e, 0).



i. Find the coordinates of the points E and F. 2 marks

ii. Find the area of the quadrilateral $EFGH$. 2 marks

iii. Find the area of the triangle QGH. 1 mark

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iv. Find an approximation for the area of the shaded region by calculating the average of the areas found in **part e.ii.** and **part e.iii.** 1 mark

v. Find the error of the approximation obtained in **part e.iv.** as a percentage of the actual area. Give your answer correct to two decimal places. 2 marks

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END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2018

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X - \mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x - \mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET