VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2019

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Letter

STUDENT NUMBER

SPECIALIST MATHEMATICS

Written examination 1

Friday 8 November 2019

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 12 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (4 marks)

Solve the differential equation $\frac{dy}{dx} = \frac{2ye^{2x}}{1+e^{2x}}$ given that $y(0) = \pi$.

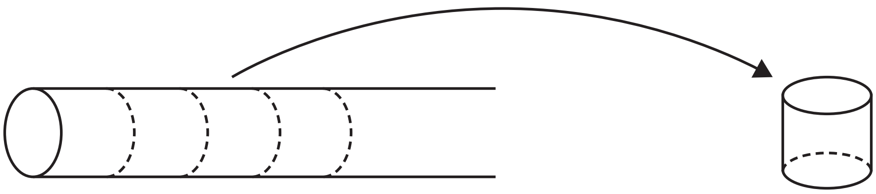
Question 2 (3 marks)

Find all values of x for which $|x-4| = \frac{x}{2} + 7$.

Question 3 (3 marks)

A machine produces chocolate in the form of a continuous cylinder of radius 0.5 cm. Smaller cylindrical pieces are cut parallel to its end, as shown in the diagram below.

The lengths of the pieces vary with a mean of 3 cm and a standard deviation of 0.1 cm.



a. Find the expected volume of a piece of chocolate in cm^3. 1 mark

b. Find the variance of the volume of a piece of chocolate in cm^6. 1 mark

c. Find the expected surface area of a piece of chocolate in cm^2. 1 mark

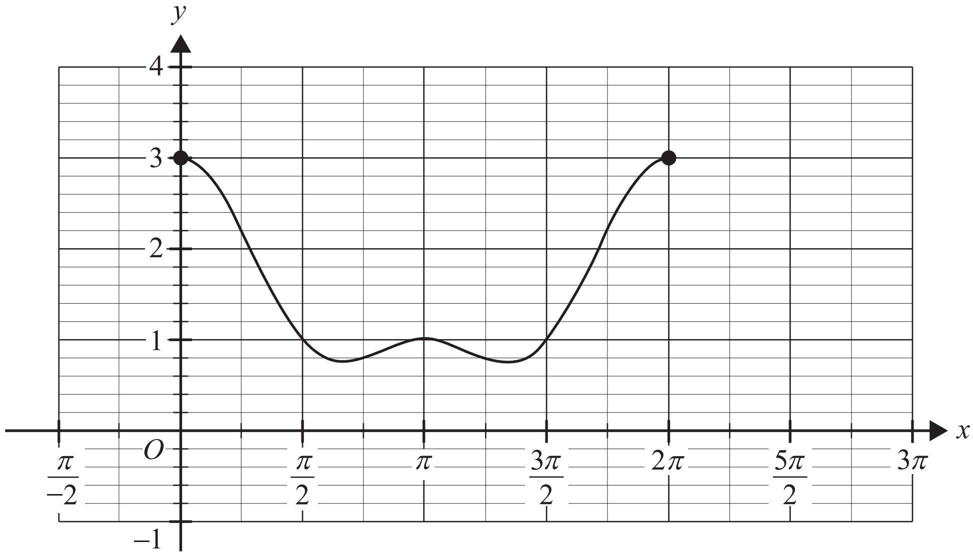
Question 4 (3 marks)

The position vectors of two particles A and B at time t seconds after they have started moving are given by $\underset{\sim}{\text{r}}_A(t) = \left(t^2 - 1\right)\underset{\sim}{\text{i}} + \left(a + \frac{t}{3}\right)\underset{\sim}{\text{j}}$ and $\underset{\sim}{\text{r}}_B(t) = \left(t^3 - t\right)\underset{\sim}{\text{i}} + \left(\arccos\left(\frac{t}{2}\right)\right)\underset{\sim}{\text{j}}$ respectively, where a is a real constant and $0 \le t \le 2$.

Find the value of a if the particles collide after they have started moving.

Question 5 (6 marks)

The graph of $f(x) = \cos^2(x) + \cos(x) + 1$ over the domain $0 \le x \le 2\pi$ is shown below.



a. **i.** Find $f'(x)$. 1 mark

ii. Hence, find the coordinates of the turning points of the graph in the interval $(0, 2\pi)$. 2 marks

b. Sketch the graph of $y = \dfrac{1}{f(x)}$ on the set of axes above. Clearly label the turning points and endpoints of this graph with their coordinates. 3 marks

Question 6 (3 marks)

Find the value of d for which the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}$ and $\underset{\sim}{c} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + d\,\underset{\sim}{k}$ are **linearly dependent**.

Question 7 (5 marks)

a. Show that $3-\sqrt{3}\,i = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$. 1 mark

b. Find $\left(3-\sqrt{3}\,i\right)^3$, expressing your answer in the form $x + iy$, where $x, y \in R$. 2 marks

c. Find the integer values of n for which $\left(3-\sqrt{3}\,i\right)^n$ is real. 1 mark

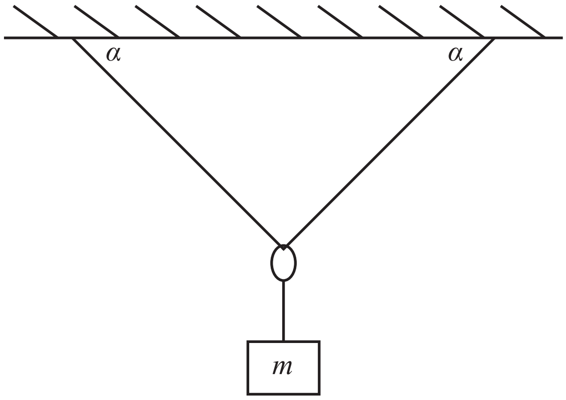
d. Find the integer values of n for which $\left(3-\sqrt{3}\,i\right)^n = ai$, where a is a real number. 1 mark

Question 8 (4 marks)

Find the volume of the solid of revolution formed when the graph of $y = \sqrt{\dfrac{1+2x}{1+x^2}}$ is rotated about the x-axis over the interval [0, 1].

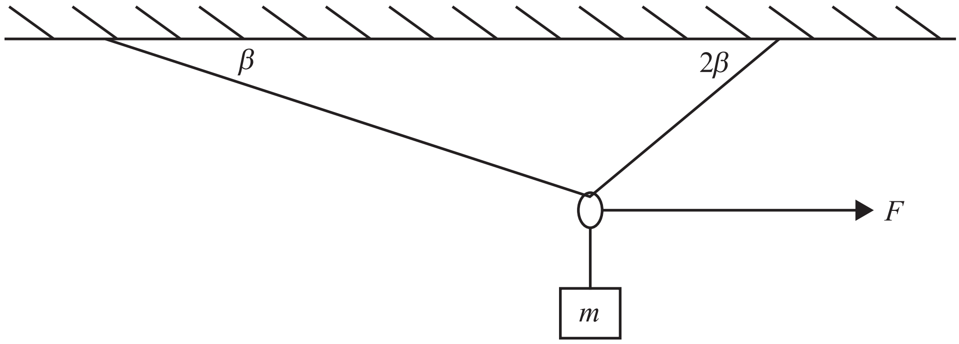
Question 9 (4 marks)

a. A light inextensible string is connected at each end to a horizontal ceiling. A mass of m kilograms hangs in equilibrium from a smooth ring on the string, as shown in the diagram below. The string makes an angle α with the ceiling.



Express the tension, T newtons, in the string in terms of m, g and α. 1 mark

b. A different light inextensible string is connected at each end to a horizontal ceiling. A mass of m kilograms hangs from a smooth ring on the string. A horizontal force of F newtons is applied to the ring. The tension in the string has a constant magnitude and the system is in equilibrium. At one end the string makes an angle β with the ceiling and at the other end the string makes an angle 2β with the ceiling, as shown in the diagram below.



Show that $F = mg\left(\frac{1-\cos(\beta)}{\sin(\beta)}\right)$. 3 marks

Question 10 (5 marks)

Find $\frac{dy}{dx}$ at the point $\left(\frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}}\right)$ for the curve defined by the relation $\sin(x^2) + \cos(y^2) = \frac{3\sqrt{2}}{\pi}xy$.

Give your answer in the form $\frac{\pi - a\sqrt{b}}{\sqrt{a}\left(\pi + \sqrt{b}\right)}$, where $a, b \in Z^+$.

END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2019

SPECIALIST MATHEMATICS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

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Specialist Mathematics formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
volume of a sphere	$\frac{4}{3}\pi r^3$
area of a triangle	$\frac{1}{2}bc\sin(A)$
sine rule	$\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	
$\sin(2x) = 2\sin(x)\cos(x)$	$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

Circular functions – continued

Function	$\sin^{-1}$ or arcsin	$\cos^{-1}$ or arccos	$\tan^{-1}$ or arctan
Domain	$[-1, 1]$	$[-1, 1]$	R
Range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r\left(\cos(\theta) + i\sin(\theta)\right) = r\,\text{cis}(\theta)$	
$\lvert z \rvert = \sqrt{x^2 + y^2} = r$	$-\pi < \text{Arg}(z) \le \pi$
$z_1 z_2 = r_1 r_2 \,\text{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}\left(\theta_1 - \theta_2\right)$
$z^n = r^n \,\text{cis}(n\theta)$ (de Moivre's theorem)	

Probability and statistics

for random variables X and Y	$\text{E}(aX + b) = a\text{E}(X) + b$ $\text{E}(aX + bY) = a\text{E}(X) + b\text{E}(Y)$ $\text{var}(aX + b) = a^2\text{var}(X)$
for independent random variables X and Y	$\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y)$
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$
distribution of sample mean $\overline{X}$	mean $\text{E}\left(\overline{X}\right) = \mu$ variance $\text{var}\left(\overline{X}\right) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$	$\int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$	$\int e^{ax}\,dx=\frac{1}{a}e^{ax}+c$
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$	$\int \frac{1}{x}\,dx=\log_e\lvert x\rvert+c$
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$	$\int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$	$\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$
$\frac{d}{dx}\left(\tan(ax)\right)=a\sec^2(ax)$	$\int \sec^2(ax)\,dx=\frac{1}{a}\tan(ax)+c$
$\frac{d}{dx}\left(\sin^{-1}(x)\right)=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+c,\ a>0$
$\frac{d}{dx}\left(\cos^{-1}(x)\right)=\frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\frac{x}{a}\right)+c,\ a>0$
$\frac{d}{dx}\left(\tan^{-1}(x)\right)=\frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2}\,dx=\tan^{-1}\left(\frac{x}{a}\right)+c$
	$\int (ax+b)^n\,dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$
	$\int (ax+b)^{-1}\,dx=\frac{1}{a}\log_e\lvert ax+b\rvert+c$
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$
Euler's method	If $\frac{dy}{dx}=f(x)$, $x_0=a$ and $y_0=b$, then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+hf(x_n)$
acceleration	$a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
arc length	$\int_{x_1}^{x_2}\sqrt{1+\left(f'(x)\right)^2}\,dx$ or $\int_{t_1}^{t_2}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt$

Vectors in two and three dimensions

$\underset{\sim}{r}=x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$
$\lvert \underset{\sim}{r}\rvert=\sqrt{x^2+y^2+z^2}=r$
$\dot{\underset{\sim}{r}}=\frac{d\underset{\sim}{r}}{dt}=\frac{dx}{dt}\underset{\sim}{i}+\frac{dy}{dt}\underset{\sim}{j}+\frac{dz}{dt}\underset{\sim}{k}$
$\underset{\sim}{r}_1\cdot\underset{\sim}{r}_2=r_1r_2\cos(\theta)=x_1x_2+y_1y_2+z_1z_2$

Mechanics

momentum	$\underset{\sim}{p}=m\underset{\sim}{v}$
equation of motion	$\underset{\sim}{R}=m\underset{\sim}{a}$

END OF FORMULA SHEET