SPECIALIST MATHEMATICS

Written examination 1

Friday 8 November 2019

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

<table>
<thead>
<tr>
<th>Number of questions</th>
<th>Number of questions to be answered</th>
<th>Number of marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 12 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your student number in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.
Instructions
Answer all questions in the spaces provided.
Unless otherwise specified, an exact answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.
Take the acceleration due to gravity to have magnitude $g$ m$s^{-2}$, where $g = 9.8$

Question 1 (4 marks)
Solve the differential equation $\frac{dy}{dx} = \frac{2ye^{2x}}{1 + e^{2x}}$ given that $y(0) = \pi$.

Question 2 (3 marks)
Find all values of $x$ for which $|x - 4| = \frac{x}{2} + 7$. 

Question 3 (3 marks)
A machine produces chocolate in the form of a continuous cylinder of radius 0.5 cm. Smaller cylindrical pieces are cut parallel to its end, as shown in the diagram below.
The lengths of the pieces vary with a mean of 3 cm and a standard deviation of 0.1 cm.

a. Find the expected volume of a piece of chocolate in cm\(^3\).  
   1 mark
   

b. Find the variance of the volume of a piece of chocolate in cm\(^6\).  
   1 mark
   

c. Find the expected surface area of a piece of chocolate in cm\(^2\).  
   1 mark
Question 4 (3 marks)
The position vectors of two particles $A$ and $B$ at time $t$ seconds after they have started moving are given by

\[ r_A(t) = (t^2 - 1) \mathbf{i} + \left( a + \frac{t}{3} \right) \mathbf{j} \quad \text{and} \quad r_B(t) = (t^3 - 1) \mathbf{i} + \left( \arccos \left( \frac{t}{2} \right) \right) \mathbf{j} \]

respectively, where $a$ is a real constant and $0 \leq t \leq 2$.

Find the value of $a$ if the particles collide after they have started moving.
Question 5 (6 marks)
The graph of \( f(x) = \cos^2(x) + \cos(x) + 1 \) over the domain \( 0 \leq x \leq 2\pi \) is shown below.

a.  
   i.  Find \( f'(x) \).  
       1 mark

   ii. Hence, find the coordinates of the turning points of the graph in the interval \( (0, 2\pi) \).  
       2 marks

b.  Sketch the graph of \( y = \frac{1}{f(x)} \) on the set of axes above. Clearly label the turning points and endpoints of this graph with their coordinates.  
    3 marks
Question 6 (3 marks)
Find the value of $d$ for which the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$ and $\mathbf{c} = -6\mathbf{i} + 2\mathbf{j} + d\mathbf{k}$ are linearly dependent.
Question 7 (5 marks)
a. Show that $3 - \sqrt{3}i = 2\sqrt{3} \text{cis} \left( -\frac{\pi}{6} \right)$.  

b. Find $(3 - \sqrt{3}i)^3$, expressing your answer in the form $x + iy$, where $x, y \in \mathbb{R}$.  

c. Find the integer values of $n$ for which $(3 - \sqrt{3}i)^n$ is real.  

d. Find the integer values of $n$ for which $(3 - \sqrt{3}i)^n = ai$, where $a$ is a real number.
Question 8 (4 marks)

Find the volume of the solid of revolution formed when the graph of \( y = \frac{\sqrt{1 + 2x}}{\sqrt{1 + x^2}} \) is rotated about the x-axis over the interval \([0, 1]\).
Question 9 (4 marks)
a. A light inextensible string is connected at each end to a horizontal ceiling. A mass of $m$ kilograms hangs in equilibrium from a smooth ring on the string, as shown in the diagram below. The string makes an angle $\alpha$ with the ceiling.

Express the tension, $T$ newtons, in the string in terms of $m$, $g$ and $\alpha$.  

1 mark
b. A different light inextensible string is connected at each end to a horizontal ceiling. A mass of $m$ kilograms hangs from a smooth ring on the string. A horizontal force of $F$ newtons is applied to the ring. The tension in the string has a constant magnitude and the system is in equilibrium. At one end the string makes an angle $\beta$ with the ceiling and at the other end the string makes an angle $2\beta$ with the ceiling, as shown in the diagram below.

Show that $F = mg\left(\frac{1 - \cos(\beta)}{\sin(\beta)}\right)$. 3 marks
Question 10 (5 marks)

Find \( \frac{dy}{dx} \) at the point \( \left( \frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}} \right) \) for the curve defined by the relation \( \sin(x^2) + \cos(y^2) = \frac{3\sqrt{2}}{\pi} \cdot xy \).

Give your answer in the form \( \frac{\pi - a\sqrt{b}}{\sqrt{a}(\pi + \sqrt{b})} \), where \( a, b \in \mathbb{Z}^+ \).
SPECIALIST MATHEMATICS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.
## Specialist Mathematics formulas

### Mensuration

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{2}(a + b)h)</td>
<td>area of a trapezium</td>
</tr>
<tr>
<td>(2\pi rh)</td>
<td>curved surface area of a cylinder</td>
</tr>
<tr>
<td>(\pi r^2 h)</td>
<td>volume of a cylinder</td>
</tr>
<tr>
<td>(\frac{1}{3}\pi r^2 h)</td>
<td>volume of a cone</td>
</tr>
<tr>
<td>(\frac{1}{3}Ah)</td>
<td>volume of a pyramid</td>
</tr>
<tr>
<td>(\frac{4}{3}\pi r^3)</td>
<td>volume of a sphere</td>
</tr>
<tr>
<td>(\frac{1}{2}bc\sin(A))</td>
<td>area of a triangle</td>
</tr>
<tr>
<td>(\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)})</td>
<td>sine rule</td>
</tr>
<tr>
<td>(c^2 = a^2 + b^2 - 2ab\cos(C))</td>
<td>cosine rule</td>
</tr>
</tbody>
</table>

### Circular functions

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\cos^2(x) + \sin^2(x) = 1)</td>
<td></td>
</tr>
<tr>
<td>(1 + \tan^2(x) = \sec^2(x))</td>
<td>(\cot^2(x) + 1 = \csc^2(x))</td>
</tr>
<tr>
<td>(\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y))</td>
<td>(\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y))</td>
</tr>
<tr>
<td>(\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y))</td>
<td>(\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y))</td>
</tr>
<tr>
<td>(\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)})</td>
<td>(\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)})</td>
</tr>
<tr>
<td>(\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x))</td>
<td></td>
</tr>
<tr>
<td>(\sin(2x) = 2\sin(x)\cos(x))</td>
<td>(\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)})</td>
</tr>
</tbody>
</table>
Circular functions – continued

<table>
<thead>
<tr>
<th>Function</th>
<th>$\sin^{-1}$ or $\arcsin$</th>
<th>$\cos^{-1}$ or $\arccos$</th>
<th>$\tan^{-1}$ or $\arctan$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>$[-1, 1]$</td>
<td>$[-1, 1]$</td>
<td>$\mathbb{R}$</td>
</tr>
<tr>
<td>Range</td>
<td>$\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$</td>
<td>$[0, \pi]$</td>
<td>$\left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$</td>
</tr>
</tbody>
</table>

Algebra (complex numbers)

<table>
<thead>
<tr>
<th>$z = x + iy = r \cos(\theta) + i \sin(\theta)$</th>
<th>$r \cos(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>z</td>
</tr>
<tr>
<td>$z_1 z_2 = r_1 r_2 \cos(\theta_1 + \theta_2)$</td>
<td>$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cos(\theta_1 - \theta_2)$</td>
</tr>
<tr>
<td>$z^n = r^n \cos(n\theta)$ (de Moivre’s theorem)</td>
<td></td>
</tr>
</tbody>
</table>

Probability and statistics

<table>
<thead>
<tr>
<th>for random variables $X$ and $Y$</th>
<th>$E(aX + b) = aE(X) + b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E(aX + bY) = aE(X) + bE(Y)$</td>
</tr>
<tr>
<td></td>
<td>$\text{var}(aX + b) = a^2 \text{var}(X)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>for independent random variables $X$ and $Y$</th>
<th>$\text{var}(aX + bY) = a^2 \text{var}(X) + b^2 \text{var}(Y)$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>approximate confidence interval for $\mu$</th>
<th>$\left( \bar{X} - z \frac{s}{\sqrt{n}}, \bar{X} + z \frac{s}{\sqrt{n}} \right)$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>distribution of sample mean $\bar{X}$</th>
<th>mean $E(\bar{X}) = \mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>variance $\text{var}(\bar{X}) = \frac{\sigma^2}{n}$</td>
</tr>
</tbody>
</table>
## Calculus

\[
\frac{d}{dx}(x^n) = nx^{n-1} \quad \int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, \quad n \neq -1
\]

\[
\frac{d}{dx}(e^{ax}) = ae^{ax} \quad \int e^{ax} \, dx = \frac{1}{a} e^{ax} + c
\]

\[
\frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad \int \frac{1}{x} \, dx = \ln|x| + c
\]

\[
\frac{d}{dx}(\sin(ax)) = a \cos(ax) \quad \int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + c
\]

\[
\frac{d}{dx}(\cos(ax)) = -a \sin(ax) \quad \int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + c
\]

\[
\frac{d}{dx}(\tan(ax)) = a \sec^2(ax) \quad \int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax) + c
\]

\[
\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \quad \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c, \quad a > 0
\]

\[
\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{a^2 - x^2}} \quad \int -\frac{1}{\sqrt{a^2 - x^2}} \, dx = \cos^{-1}\left(\frac{x}{a}\right) + c, \quad a > 0
\]

\[
\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} \quad \int \frac{1}{1+x^2} \, dx = \tan^{-1}\left(\frac{x}{a}\right) + c
\]

\[
(ax + b)^n \, dx = \frac{1}{a(n+1)} (ax + b)^{n+1} + c, \quad n \neq -1
\]

\[
\int (ax + b)^{-1} \, dx = \frac{1}{a} \log_e|ax + b| + c
\]

**product rule**

\[
\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}
\]

**quotient rule**

\[
\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]

**chain rule**

\[
\frac{dv}{dx} = \frac{dy}{du} \frac{du}{dx}
\]

**Euler’s method**

If \( \frac{dy}{dx} = f(x) \), \( x_0 = a \) and \( y_0 = b \), then \( x_{n+1} = x_n + h \) and \( y_{n+1} = y_n + hf(x_n) \)

**acceleration**

\[
a = \frac{d^2x}{dt^2} = \frac{dv}{dx} = v \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2}v^2\right)
\]

**arc length**

\[
\int_s^t \sqrt{1 + (f''(x))^2} \, dx \quad \text{or} \quad \int_s^t \sqrt{(x'(t))^2 + (y'(t))^2} \, dt
\]

## Vectors in two and three dimensions

\[
\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}
\]

\[
|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = r
\]

\[
\dot{\mathbf{r}} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k}
\]

\[
\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos(\theta) = x_1 x_2 + y_1 y_2 + z_1 z_2
\]

## Mechanics

**momentum**

\[
p = mv
\]

**equation of motion**

\[
R = ma
\]

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**END OF FORMULA SHEET**