VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2021

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Letter

STUDENT NUMBER

MATHEMATICAL METHODS

Written examination 2

Thursday 4 November 2021

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	5	5	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 26 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The period of the function with rule $y = \tan\left(\frac{\pi x}{2}\right)$ is

A. 1
B. 2
C. 4
D. 2π
E. 4π

Question 2

The graph of $y = \log_e(x) + \log_e(2x)$, where $x > 0$, is identical, over the same domain, to the graph of

A. $y = 2\log_e\left(\frac{1}{2}x\right)$
B. $y = 2\log_e(2x)$
C. $y = \log_e(2x^2)$
D. $y = \log_e(3x)$
E. $y = \log_e(4x)$

Question 3

A box contains many coloured glass beads.

A random sample of 48 beads is selected and it is found that the proportion of blue-coloured beads in this sample is 0.125

Based on this sample, a 95% confidence interval for the proportion of blue-coloured glass beads is

A. (0.0314, 0.2186)
B. (0.0465, 0.2035)
C. (0.0018, 0.2482)
D. (0.0896, 0.1604)
E. (0.0264, 0.2136)

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Question 4

The maximum value of the function $h : [0, 2] \to R, h(x) = (x - 2)e^x$ is

A. $-e$
B. 0
C. 1
D. 2
E. e

Question 5

Consider the following four functional relations.

$$f(x) = f(-x) \qquad -f(x) = f(-x) \qquad f(x) = -f(x) \qquad (f(x))^2 = f(x^2)$$

The number of these functional relations that are satisfied by the function $f: R \to R, f(x) = x$ is

A. 0
B. 1
C. 2
D. 3
E. 4

Question 6

The probability of winning a game is 0.25

The probability of winning a game is independent of winning any other game.

If Ben plays 10 games, the probability that he will win exactly four times is closest to

A. 0.1460
B. 0.2241
C. 0.9219
D. 0.0781
E. 0.7759

Question 7

The tangent to the graph of $y = x^3 - ax^2 + 1$ at $x = 1$ passes through the origin.

The value of a is

A. $\frac{1}{2}$

B. 1

C. $\frac{3}{2}$

D. 2

E. $\frac{5}{2}$

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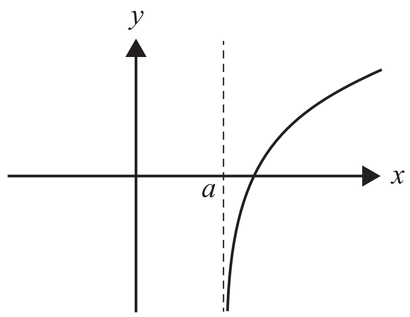
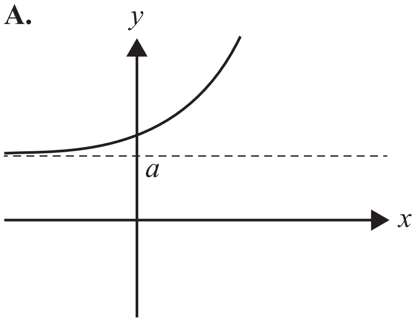
Question 8

The graph of the function f is shown below.

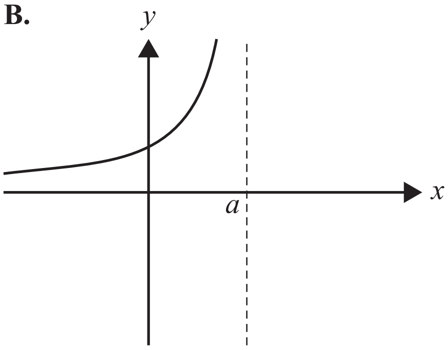


The graph corresponding to f' is

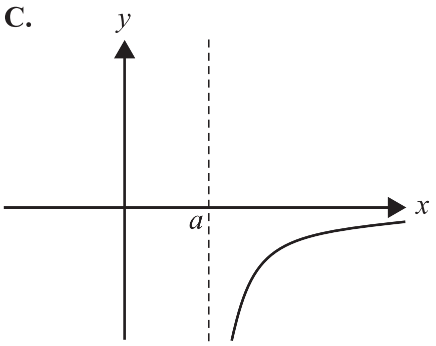
A.



B.



C.



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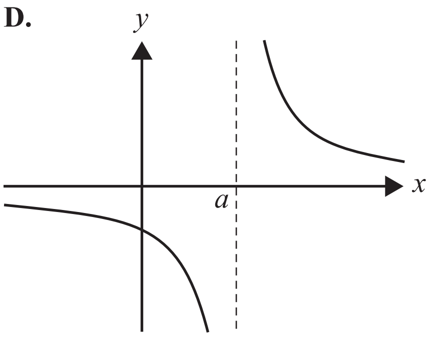
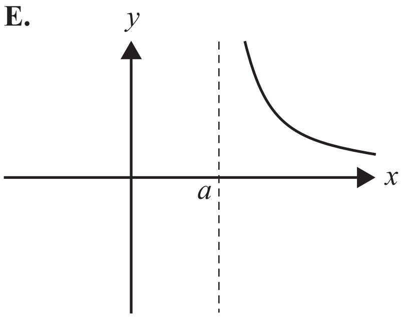


E.



Question 9

Let $g(x) = x + 2$ and $f(x) = x^2 - 4$.

If h is the composite function given by $h : [-5, -1) \to R$, $h(x) = f\big(g(x)\big)$, then the range of h is

A. $(-3, 5]$

B. $[-3, 5)$

C. $(-3, 5)$

D. $(-4, 5]$

E. $[-4, 5]$

Question 10

Consider the functions $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{1-2x}$, defined over their maximal domains.

The maximal domain of the function $h = f + g$ is

A. $\left(-2, \dfrac{1}{2}\right)$

B. $[-2, \infty)$

C. $(-\infty, -2) \cup \left(\dfrac{1}{2}, \infty\right)$

D. $\left[-2, \dfrac{1}{2}\right]$

E. $[-2, 1]$

Question 11

If $\int_0^a f(x)dx = k$, then $\int_0^a \big(3f(x) + 2\big)dx$ is

A. $3k + 2a$

B. $3k$

C. $k + 2a$

D. $k + 2$

E. $3k + 2$

Question 12

For a certain species of bird, the proportion of birds with a crest is known to be $\dfrac{3}{5}$.

Let $\hat{P}$ be the random variable representing the proportion of birds with a crest in samples of size n for this specific bird.

The smallest sample size for which the standard deviation of $\hat{P}$ is less than 0.08 is

A. 7

B. 27

C. 37

D. 38

E. 43

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Question 13

The value of an investment, in dollars, after n months can be modelled by the function

$$f(n) = 2500 \times (1.004)^n$$

where $n \in \{0, 1, 2, \ldots\}$.

The average rate of change of the value of the investment over the first 12 months is closest to

A. \$10.00 per month.

B. \$10.20 per month.

C. \$10.50 per month.

D. \$125.00 per month.

E. \$127.00 per month.

Question 14

A value of k for which the average value of $y = \cos\left(kx - \frac{\pi}{2}\right)$ over the interval $[0, \pi]$ is equal to the average value of $y = \sin(x)$ over the same interval is

A. $\frac{1}{6}$

B. $\frac{1}{5}$

C. $\frac{1}{4}$

D. $\frac{1}{3}$

E. $\frac{1}{2}$

Question 15

Four fair coins are tossed at the same time.

The outcome for each coin is independent of the outcome for any other coin.

The probability that there is an equal number of heads and tails, given that there is at least one head, is

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{3}{4}$

D. $\frac{2}{5}$

E. $\frac{4}{7}$

Question 16

Let $\cos(x) = \dfrac{3}{5}$ and $\sin^2(y) = \dfrac{25}{169}$, where $x \in \left[\dfrac{3\pi}{2},\ 2\pi\right]$ and $y \in \left[\dfrac{3\pi}{2},\ 2\pi\right]$.

The value of $\sin(x) + \cos(y)$ is

A. $\dfrac{8}{65}$

B. $-\dfrac{112}{65}$

C. $\dfrac{112}{65}$

D. $-\dfrac{8}{65}$

E. $\dfrac{64}{65}$

Question 17

A discrete random variable X has a binomial distribution with a probability of success of $p = 0.1$ for n trials, where $n > 2$.

If the probability of obtaining at least two successes after n trials is at least 0.5, then the smallest possible value of n is

A. 15

B. 16

C. 17

D. 18

E. 19

Question 18

Let $f: R \to R$, $f(x) = (2x - 1)(2x + 1)(3x - 1)$ and $g: (-\infty, 0) \to R$, $g(x) = x\log_e(-x)$.

The maximum number of solutions for the equation $f(x - k) = g(x)$, where $k \in R$, is

A. 0

B. 1

C. 2

D. 3

E. 4

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Question 19

Which one of the following functions is differentiable for all real values of x?

A. $f(x)=\begin{cases} x & x<0 \\ -x & x\geq 0 \end{cases}$

B. $f(x)=\begin{cases} x & x<0 \\ -x & x>0 \end{cases}$

C. $f(x)=\begin{cases} 8x+4 & x<0 \\ (2x+1)^2 & x\geq 0 \end{cases}$

D. $f(x)=\begin{cases} 2x+1 & x<0 \\ (2x+1)^2 & x\geq 0 \end{cases}$

E. $f(x)=\begin{cases} 4x+1 & x<0 \\ (2x+1)^2 & x\geq 0 \end{cases}$

Question 20

Let A and B be two independent events from a sample space.

If $\Pr(A) = p$, $\Pr(B) = p^2$ and $\Pr(A) + \Pr(B) = 1$, then $\Pr(A' \cup B)$ is equal to

A. $1 - p - p^2$

B. $p^2 - p^3$

C. $p - p^3$

D. $1 - p + p^3$

E. $1 - p - p^2 + p^3$

END OF SECTION A
TURN OVER

SECTION B

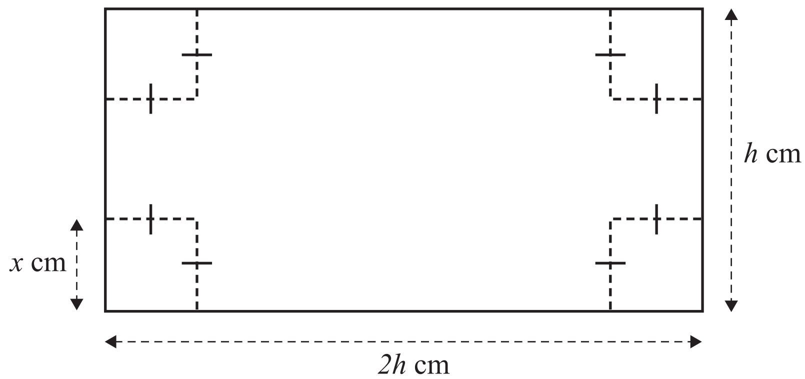
Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (14 marks)

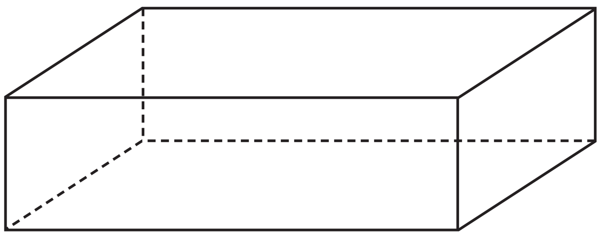
A rectangular sheet of cardboard has a width of h centimetres. Its length is twice its width.

Squares of side length x centimetres, where $x > 0$, are cut from each of the corners, as shown in the diagram below.



The sides of this sheet of cardboard are then folded up to make a rectangular box with an open top, as shown in the diagram below.

Assume that the thickness of the cardboard is negligible and that $V_{box} > 0$.



A box is to be made from a sheet of cardboard with $h = 25$ cm.

a. Show that the volume, V_{box}, in cubic centimetres, is given by $V_{box}(x) = 2x(25 - 2x)(25 - x)$. 1 mark

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b. State the domain of V_{box}. 1 mark

c. Find the derivative of V_{box} with respect to x. 1 mark

d. Calculate the maximum possible volume of the box and for which value of x this occurs. 3 marks

e. Waste minimisation is a goal when making cardboard boxes.

Percentage wasted is based on the area of the sheet of cardboard that is cut out before the box is made.

Find the percentage of the sheet of cardboard that is wasted when $x = 5$. 2 marks

Now consider a box made from a rectangular sheet of cardboard where $h > 0$ and the box's length is still twice its width.

f. **i.** Let V_{box} be the function that gives the volume of the box.

State the domain of V_{box} in terms of h. 1 mark

ii. Find the maximum volume for any such rectangular box, V_{box}, in terms of h. 3 marks

g. Now consider making a box from a square sheet of cardboard with side lengths of h centimetres.

Show that the maximum volume of the box occurs when $x = \frac{h}{6}$. 2 marks

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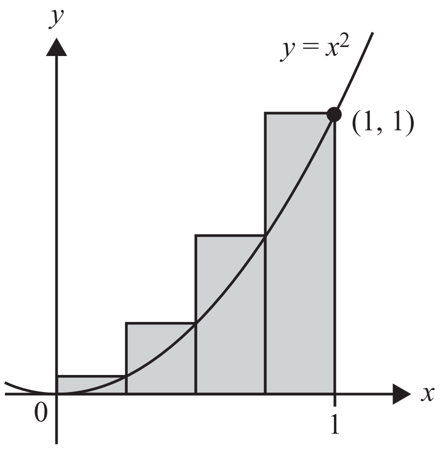
Question 2 (10 marks)

Four rectangles of equal width are drawn and used to approximate the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$.

The heights of the rectangles are the values of the graph of $y = x^2$ at the right endpoint of each rectangle, as shown in the graph below.



a. State the width of each of the rectangles shown above. 1 mark

b. Find the total area of the four rectangles shown above. 1 mark

c. Find the area between the graph of $y = x^2$, the x-axis and the line $x = 1$. 2 marks

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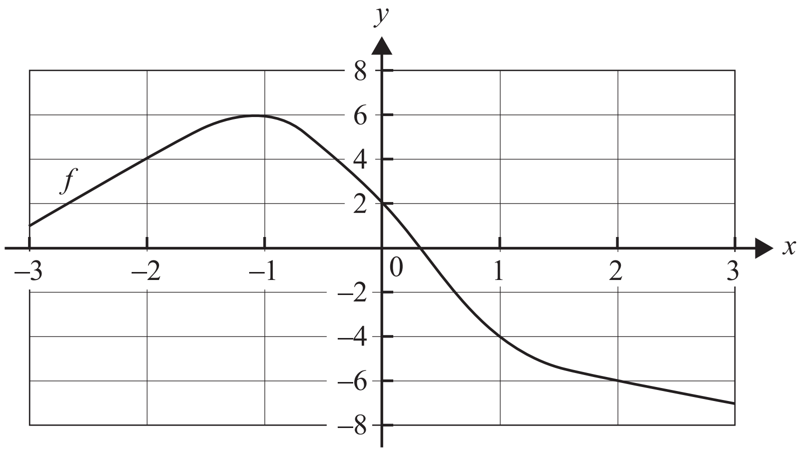
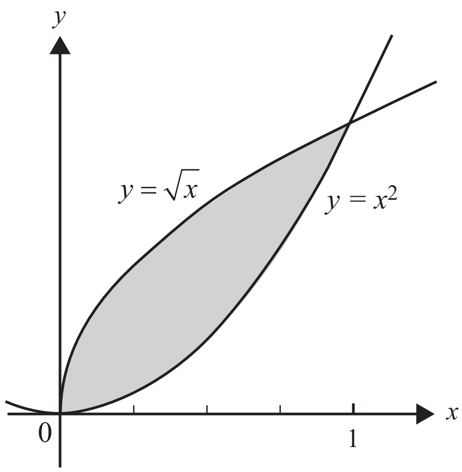
d. The graph of f is shown below.



Approximate $\int_{-2}^{2} f(x)\,dx$ using four rectangles of equal width and the right endpoint of each rectangle. 1 mark

Parts of the graphs of $y = x^2$ and $y = \sqrt{x}$ are shown below.



e. Find the area of the shaded region. 1 mark

f. The graph of $y = x^2$ is transformed to the graph of $y = ax^2$, where $a \in (0, 2]$.

Find the values of a such that the area defined by the region(s) bounded by the graphs of $y = ax^2$ and $y = \sqrt{x}$ and the lines $x = 0$ and $x = a$ is equal to $\frac{1}{3}$. Give your answer correct to two decimal places. 4 marks

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Question 3 (12 marks)

Let $q(x) = \log_e(x^2 - 1) - \log_e(1 - x)$.

a. State the maximal domain and the range of q. 2 marks

b. i. Find the equation of the tangent to the graph of q when $x = -2$. 1 mark

ii. Find the equation of the line that is perpendicular to the graph of q when $x = -2$ and passes through the point $(-2, 0)$. 1 mark

Let $p(x) = e^{-2x} - 2e^{-x} + 1$.

c. Explain why p is not a one-to-one function. 1 mark

d. Find the gradient of the tangent to the graph of p at $x = a$. 1 mark

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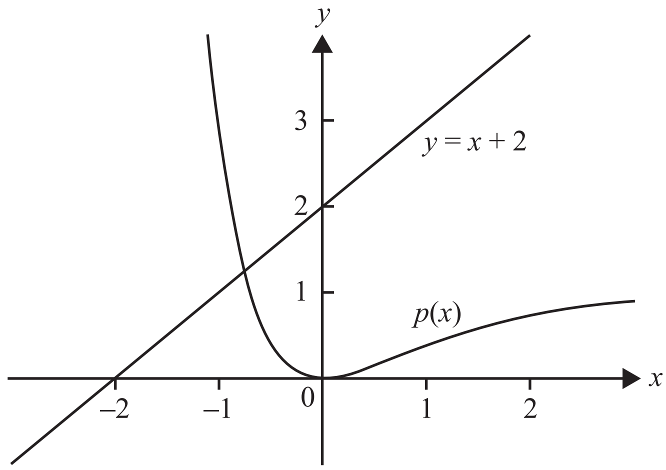
The diagram below shows parts of the graph of p and the line $y = x + 2$.



The line $y = x + 2$ and the tangent to the graph of p at $x = a$ intersect with an acute angle of θ between them.

e. Find the value(s) of a for which $\theta = 60°$. Give your answer(s) correct to two decimal places. 3 marks

f. Find the x-coordinate of the point of intersection between the line $y = x + 2$ and the graph of p, and hence find the area bounded by $y = x + 2$, the graph of p and the x-axis, both correct to three decimal places. 3 marks

Question 4 (14 marks)

A teacher coaches their school's table tennis team.

The teacher has an adjustable ball machine that they use to help the players practise.

The speed, measured in metres per second, of the balls shot by the ball machine is a normally distributed random variable W.

The teacher sets the ball machine with a mean speed of 10 metres per second and a standard deviation of 0.8 metres per second.

a. Determine $\Pr(W \geq 11)$, correct to three decimal places. 1 mark

b. Find the value of k, in metres per second, which 80% of ball speeds are below. Give your answer in metres per second, correct to one decimal place. 1 mark

The teacher adjusts the height setting for the ball machine. The machine now shoots balls high above the table tennis table.

Unfortunately, with the new height setting, 8% of balls do not land on the table.

Let $\hat{P}$ be the random variable representing the sample proportion of balls that do not land on the table in random samples of 25 balls.

c. Find the mean and the standard deviation of $\hat{P}$. 2 marks

d. Use the binomial distribution to find $\Pr(\hat{P} > 0.1)$, correct to three decimal places. 2 marks

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The teacher can also adjust the spin setting on the ball machine.

The spin, measured in revolutions per second, is a continuous random variable X with the probability density function

$$f(x) = \begin{cases} \dfrac{x}{500} & 0 \le x < 20 \\ \dfrac{50-x}{750} & 20 \le x \le 50 \\ 0 & \text{elsewhere} \end{cases}$$

e. Find the maximum possible spin applied by the ball machine, in revolutions per second. 1 mark

f. Find the median spin, in revolutions per second, correct to one decimal place. 2 marks

g. Find the standard deviation of the spin, in revolutions per second, correct to one decimal place. 3 marks

h. The teacher adjusts the spin setting so that the median spin becomes 30 revolutions per second. This will transform the original probability density function f to a new probability density function g, where $g(x) = af\left(\frac{x}{b}\right)$.

Find the values of a and b for which the new median spin is 30 revolutions per second, giving your answer correct to two decimal places. 2 marks

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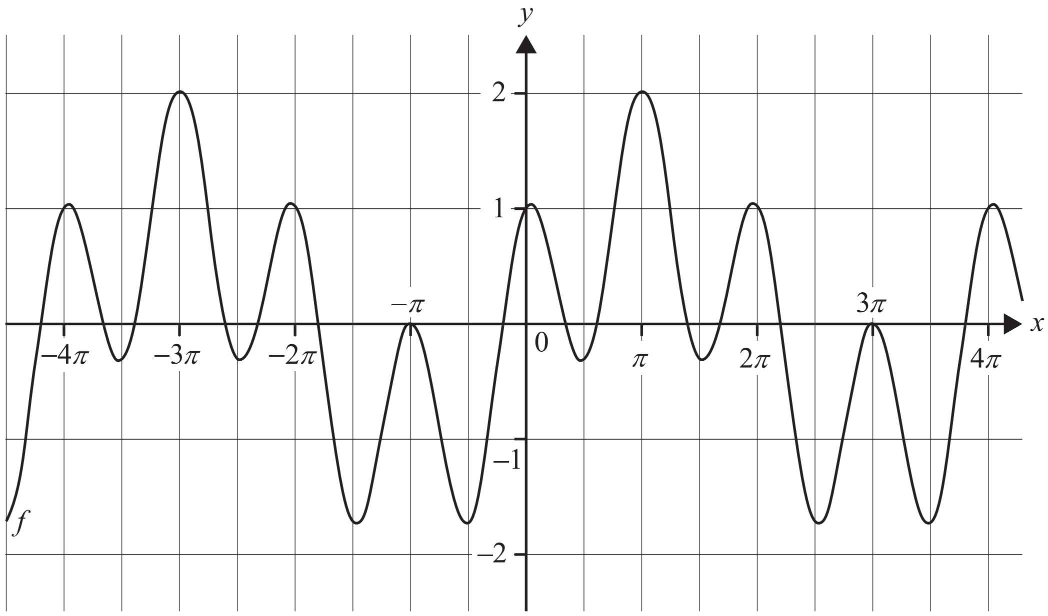
Question 5 (10 marks)

Part of the graph of $f: R \to R,\ f(x) = \sin\left(\frac{x}{2}\right) + \cos(2x)$ is shown below.



a. State the period of *f*. 1 mark

b. State the minimum value of *f*, correct to three decimal places. 1 mark

c. Find the smallest positive value of *h* for which $f(h - x) = f(x)$. 1 mark

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Consider the set of functions of the form $g_a : R \to R$, $g_a(x) = \sin\left(\frac{x}{a}\right) + \cos(ax)$, where a is a positive integer.

d. State the value of a such that $g_a(x) = f(x)$ for all x. 1 mark

e. **i.** Find an antiderivative of g_a in terms of a. 1 mark

ii. Use a definite integral to show that the area bounded by g_a and the x-axis over the interval $[0, 2a\pi]$ is equal above and below the x-axis for all values of a. 3 marks

f. Explain why the maximum value of g_a cannot be greater than 2 for all values of a and why the minimum value of g_a cannot be less than –2 for all values of a. 1 mark

g. Find the greatest possible minimum value of g_a. 1 mark

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END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2021

MATHEMATICAL METHODS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET