2006 Mathematical Methods (CAS) GA 3: Examination 2

GENERAL COMMENTS

There were 539 students who sat this examination in 2006. Marks ranged from 4 to the maximum possible score of 80. Student responses showed that the paper was accessible and that it provided an opportunity for students to demonstrate what they knew.

Of the whole cohort, 12% of students scored 90% or more of the available marks, and 60% scored 50% or more of the available marks. The mean score for the paper was 45.5, with means of 14.4 (out of 22) for the multiple-choice section and of 30.1 (out of 58) for the extended-answer section. The median score for the paper was 46 marks. Only 7% of students scored 20% or less of the available marks.

Generally, the symbolic facility of CAS was used well. There was no discernable advantage seen by the assessors of one CAS over another, although in some cases techniques for dealing with the mathematics involved varied according to the CAS. There were many very good responses to the questions in Section 2 and several students were able to work through the questions completely and obtain full marks (or close to it) for this section.

There was evidence to suggest that some students spent too much time on the multiple-choice questions in Section 1 and were therefore not able to make a reasonable attempt at Section 2. Students should be encouraged to balance the amount of time they spend on each section of the paper with respect to the total marks available for that section.

Students sometimes did not give answers in exact form when this was explicitly asked for. Students must ensure that they do not give numerical approximations when an exact answer is required. If an exact answer is required, students should think carefully about the method they should use to arrive at the exact answer.

Students must also ensure that they show their working. If a question is worth more than one mark, they risk losing all available marks if only the answer is given, and it is incorrect. The instructions at the beginning of the paper state that if more than one mark is available for a question then appropriate working **must** be shown.

Students lost marks when they:

- did not answer the question asked
- gave decimal answers when exact answers were required
- gave the wrong number of decimal places or rounded incorrectly.

When students present working and develop their solutions, they are expected to use conventional mathematical expressions, symbols, notation and terminology. This was generally well done.

SPECIFIC INFORMATION

Section 1

The table below indicates the percentage of students who chose each option. The correct answer is indicated by shading.

Question	% A	% B	% C	% D	% E	% No Answer	Comments
1	1	2	92	5	0	0	
2	2	84	6	7	1	0	
3	71	12	3	4	9	0	
4	24	59	4	5	8	0	
5	2	6	17	68	6	1	
6	8	6	4	74	7	0	
7	6	60	9	2	23	0	
8	4	13	16	59	9	0	
9	5	84	4	2	5	0	

Question	% A	% B	% C	% D	% E	% No Answer	Comments
10	48	12	24	9	6	1	Since the volume *V* of a sphere of radius *r* is given by $V = \frac{4}{3}\pi r^3$, the rate of change of the volume is given by $\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$. Now $\frac{dr}{dt} = 3$ cm/min, so when *r* = 6 cm, $\frac{dV}{dt} = 4 \times \pi \times 6^2 \times 3 = 432\pi$ cm^3/min.
11	16	2	4	75	4	0	
12	13	8	71	3	4	0	
13	33	25	17	5	20	0	Students were required to find the rule for a transformation which maps the curve with equation $y = \log_e(x)$ to $y = \log_e(2x - 4) + 3$ or, equivalently, to $y - 3 = \log_e(2(x - 2))$. This second form shows that the transformation consists of a dilation from the *y*-axis by a factor of 0.5 followed by a shift of two units to the right and three units up. Hence the rule could be $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
14	69	14	6	5	6	0	
15	1	1	8	81	9	0	
16	66	6	19	3	5	0	
17	16	18	17	37	10	1	If $f(x) = 2x$, then $f\left(\frac{x+y}{2}\right) = 2\left(\frac{x+y}{2}\right) = x + y$. Moreover, $\frac{f(x) + f(y)}{2} = \frac{2x + 2y}{2} = x + y$. Hence the function $f(x) = 2x$ satisfies the given functional equation.
18	8	6	7	76	3	0	
19	21	17	15	11	35	1	To have either no solutions, or infinitely many solutions, the ratio of the coefficients of the *x* and the *y* terms must be equal, hence $\frac{m-2}{2} = \frac{3}{m-3}, m \neq 3$. (If *m* = 3, then the equations will have a unique solution *x* = 1 and $y = \frac{5}{3}$.) This can be rearranged to form the quadratic equation $(m - 2)(m - 3) = 6$, or $m^2 - 5m = 0$, which has solutions *m* = 0 or *m* = 5. If *m* = 0, the simultaneous equations are $-2x + 3y = 6$ and $2x - 3y = -1$ and they have no solution as the equations correspond to distinct parallel lines. If *m* = 5, the simultaneous equations are $3x + 3y = 6$ and $2x + 2y = 4$, and they have many solutions since each equation represents the same line.
20	28	3	14	4	50	1	
21	2	4	9	75	10	1	
22	14	6	69	11	0	0	

The correct response was selected by less than 50 per cent of students for only Questions 10, 13, 17 and 19.

Section 2

Question 1

1ai.

Marks	0	1	Average
%	10	90	**0.9**

$f'(x) = 2\cos(x)$

This question was well done. A common error was incorrect differentiation, giving $f'(x) = -2\cos(x)$.

1aii.

Marks	0	1	2	Average
%	20	24	56	**1.4**

Maximum value is 2, minimum is 0

Many students found the maximum value correctly; however, some students ignored the absolute value sign and gave the minimum as −2. Some tried to find the minimum using a graphical or numerical method and gave an answer such as 7.86×10^{-12} or, worse, 7.86E–12. Some students gave the coordinates of the local maxima and minima, which did not directly answer the question posed.

1bi.

Marks	0	1	Average
%	35	65	**0.7**

$\frac{5\pi}{3}$

This question was not well done. A common error was giving a solution outside the domain, usually $\frac{-\pi}{3}$, or a numerical approximation, even though the question specifically asked for an exact value.

1bii.

Marks	0	1	2	Average
%	26	19	55	**1.4**

At $x = \frac{\pi}{3}$, $y = 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}$

Hence the equation of the tangent is $y - \sqrt{3} = x - \frac{\pi}{3}$ or $y = x - \frac{\pi}{3} + \sqrt{3}$

Some students were unable to evaluate $2\sin\left(\frac{\pi}{3}\right)$ correctly, either giving an approximate answer or an incorrect one.

Some students did not write an equation. Some appeared to have used their calculator to obtain a numerical approximation to this line, such as $y = x - 0.685$. Some students found the equation of the normal.

1biii.

Marks	0	1	2	3	Average
%	29	6	10	55	**2.0**

If $y = 0$, $x = \frac{\pi}{3} - \sqrt{3}$ and so the x-axis intercept is $\left(\frac{\pi}{3} - \sqrt{3}, 0\right)$

If $x = 0$, $y = \sqrt{3} - \frac{\pi}{3}$ and so the y-axis intercept is $\left(0, \sqrt{3} - \frac{\pi}{3}\right)$

Some students who obtained the correct equation in part bii. gave numerical approximations to the intercepts, even though exact values were specified. A few students wrote the intercepts down with the coordinates reversed, for example, $\left(0, \frac{\pi}{3} - \sqrt{3}\right)$ for the *x*-axis intercept.

1c.

Marks	0	1	2	Average
%	71	12	17	**0.5**

The tangent to the curve at *A* cuts the *x*-axis at $x = \frac{\pi}{3} - \sqrt{3}$. Since the period of the function is 2π,

$m = 2\pi + 2\left(\sqrt{3} - \frac{\pi}{3}\right) = 2\sqrt{3} + \frac{4\pi}{3}$.

Another method of obtaining the correct solution was to determine that the equation of the tangent at $B\left(\frac{5\pi}{3}, -\sqrt{3}\right)$ was $y = x - \frac{5\pi}{3} - \sqrt{3}$. This cuts the *x*-axis at $\left(\frac{5\pi}{3} + \sqrt{3}, 0\right)$. Since the tangent to the curve at *A* cuts the *x*-axis at $\left(\frac{\pi}{3} - \sqrt{3}, 0\right)$, *m* will equal the difference of the *x*-intercepts, $\left(\frac{5\pi}{3} + \sqrt{3}\right) - \left(\frac{\pi}{3} - \sqrt{3}\right) = \frac{4\pi}{3} + 2\sqrt{3}$.

Some students who used a correct method gave a negative value for *m*. Some students simply took the difference of the *x*-values of points *A* and *B*, obtaining $\frac{4\pi}{3}$. Consideration of the diagram should have indicated that this was an incorrect answer. Overall this question was not well done.

1d.

Marks	0	1	2	Average
%	54	31	15	**0.7**

Solve $|\sin(x)| = 0.5$ over the interval $[0, \pi]$ since period of $f(x) = |\sin(x)|$ is π. The solutions are $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

The general solution is then $x = \frac{\pi}{6} + n\pi$ or $x = \frac{5\pi}{6} + n\pi, n \in Z$ or, equivalently, $x = n\pi \pm \frac{\pi}{6}, n \in Z$

Some students were unable to find even one solution, which was disappointing given access to CAS. Some failed to give the general form of the solution, and some used *R* (corresponding to the set of real numbers) or *J* instead of *Z*. Some students interpreted the period as 2π rather than π, suggesting that they had ignored the absolute value part. Some students were unable to make coherent interpretations of the output from CAS and wrote things such as 6.28319@n1+2.61799; in the first instance this is not likely to be an exact form, and '@n1' should be written as a single parameter, such as *n*, with the values it can take specified.

Question 2

2ai.

Marks	0	1	Average
%	46	54	**0.6**

$0.4 \times 0.3 \times 0.3 = 0.036$

This question was not well done. A common error was trying to use a transition matrix formulation to give the answer.

2aii.

Marks	0	1	2	Average
%	52	12	36	**0.9**

The possibilities are GPP, PGP and PPG (where 'G' = gym and 'P' = pool) with corresponding probabilities $0.6 \times 0.4 \times 0.3$, $0.4 \times 0.7 \times 0.4$ and $0.4 \times 0.3 \times 0.7$ respectively. The answer is $0.6 \times 0.4 \times 0.3 + 0.4 \times 0.7 \times 0.4 + 0.4 \times 0.3 \times 0.7 = 0.268$.

This was best done by constructing a tree diagram, with appropriate probabilities written on it. Again, some students tried to use a transition matrix formulation to obtain an answer. Some students only considered one possibility, such as GPP.

2b.

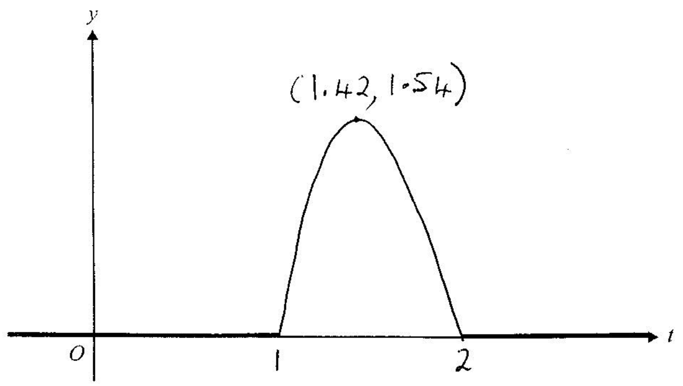
Marks	0	1	Average
%	54	46	**0.5**

In the long term, she goes to the pool on 0.364 of nights.

This part was not well done. The transition matrix $\begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$ should have been used. Students could have either directly written down the long term probability of going to the pool as $\frac{0.4}{0.4+0.7} \approx 0.364$, or else evaluated the transition matrix to a high power and selected the correct entry from it. A common error was incorrect rounding of 0.3636 to three decimal places. Students who used the first method often gave the answer as $\frac{0.7}{0.4+0.7} \approx 0.636$, which gives the long term proportion of nights she goes to the gym. Finally, some students correctly wrote down the steady state vector, but then did not indicate which number was the required proportion.

2c.

Marks	0	1	2	3	Average
%	9	10	15	66	**2.5**



This was generally well done. Errors included extending the cubic part beyond the domain of [1, 2], incorrect rounding of the coordinates of the turning point, incorrect skew indicated on the part of graph with domain [1, 2], failing to indicate what happens to the graph outside the domain [1, 2] and incorrect labeling of the points 1 and 2 on the horizontal axis (including writing '1' at the marked origin).

2d.

Marks	0	1	2	Average
%	38	5	57	**1.3**

$\int_1^{1.25} (4t^3 - 24t^2 + 44t - 24)dt \approx 0.191$ or $\int_1^{1.25} f(t)dt \approx 0.191$ (since the function f had been defined in the question). The probability, correct to three decimal places, that she spends less than 75 minutes working out when she goes to the gym is 0.191.

Many students attempted this question but, unfortunately, a common error was incorrectly converting 75 minutes as 1.15 hours. Once formulated correctly, this question could be readily done with technology. Note that it is important for the 'dt' to appear as part of the definite integral for correct mathematical notation.

2e.

Marks	0	1	2	Average
%	54	11	35	**0.9**

Consider the random variable X = number of times she spends more than 75 minutes working out on her next five visits. This random variable has a binomial distribution, with $n = 5$, $p = 0.809$ (correct to three decimal places), $\Pr(X = 4) = 0.41$ (correct to two decimal places). Hence the probability, correct to two decimal places, that she spends more than 75 minutes working out on four out of the next five times she goes to the gym is 0.41.

Alternatively, consider the random variable Y = number of times she spends less than 75 minutes working out on her next five visits. This random variable also has a binomial distribution with $n = 5$, $p = 0.191$, $\Pr(Y = 1) = 0.41$

Many students did not identify the use of the binomial as the relevant distribution. This was another question which, once set up correctly, was easily done using technology.

2f.

Marks	0	1	2	3	Average
%	40	17	11	33	**1.4**

Solve $\int_1^{M} (4t^3 - 24t^2 + 44t - 24)dt = 0.5$ for M, the median.

$M \approx 1.4588$ hours = 88 minutes (to the nearest minute)

Some students found the mean or the mode rather than the median. Some students gave a solution to a polynomial equation outside the required domain. There were a few students with rounding errors. This was another question, which once set up correctly, was easily done using technology.

Question 3

3a.

Marks	0	1	Average
%	15	85	**0.9**

When $x = 0$, $y = 3 - e^0 - e^{-0} = 1$. Hence as b is the y-intercept, $b = 1$

This was quite well done.

3b.

Marks	0	1	Average
%	34	66	**0.7**

$a = \log_e\left(\frac{3+\sqrt{5}}{2}\right)$

Students had to solve the equation $3 - e^x - e^{-x} = 0$ exactly, which gives two solutions $x = \log_e\left(\frac{3 \pm \sqrt{5}}{2}\right)$, corresponding to the two axis intercepts. The main errors were giving an approximate instead of an exact value for *a* or incorrectly writing down the value from the CAS.

3ci.

Marks	0	1	Average
%	26	74	**0.8**

x	**–0.5**	**0**	**0.5**
y	0.74	1.00	0.74

Some students had trouble rounding the value of *y* correctly when *x* = 0.5. They obtained 0.7447 to four decimal places, and some incorrectly wrote 0.75; some wrote 0.745, which is correct if required to three decimal places, but does not round to the correct value when two decimal places are required.

3cii.

Marks	0	1	2	Average
%	26	13	61	**1.4**

Using the table, the approximation for the area, in square kilometres, is $0.5 \times (1 + 2 \times 0.74) = 1.24$. Hence the area is approximately 1.2 square kilometres, correct to one decimal place.

Some students simply multiplied the *x*-values by the *y*-values and summed these to determine the area.

3ciii.

Marks	0	1	Average
%	49	51	**0.6**

$V = 1.2 \times w \times m$

This was a very disappointing result for a relatively simple formulation question. Of students who made an attempt at the question, some omitted the 1.2, while others added the *w* and *m*.

3di.

Marks	0	1	Average
%	34	66	**0.7**

$y = 1 - x^2$

Some students simply wrote $1 - x^2$, which is not an equation as required, and so were not awarded the mark.

3dii.

Marks	0	1	2	Average
%	26	11	63	**1.5**

$$\int_{-1}^{1} (1 - x^2)\,dx \approx 1.33 \text{ or } 2\int_{0}^{1} (1 - x^2)\,dx \approx 1.33$$

The area is approximately 1.33 square kilometres.

Generally, students who were correct with part di. had no trouble with this part. Again, it was important for the '*dx*' to be written as part of the definite integral.

3e.

Marks	0	1	2	3	4	Average
%	66	15	4	3	13	**0.8**

Solve $3-ke^{x}-e^{-x}=0$ for x

$$x=\log_e\left(\frac{-3\pm\sqrt{9-4k}}{-2k}\right)$$

It is necessary to check that the argument of the logarithm function is real and positive. To be real, $9 - 4k \geq 0$, so $k \leq \frac{9}{4}$.

Since it is given that $k > 0$, $0\leq\sqrt{9-4k}<3$ and so the argument of the logarithm function will always be positive.

Hence for $0 < k \leq \frac{9}{4}$, the equation $3 - ke^{x} - e^{-x} = 0$ has one or two (real) solutions for x.

This question was not well done, with only a few students making substantial progress. Some students attempted to apply the formula for the discriminant of a quadratic equation to the equation $3 - ke^{x} - e^{-x} = 0$ directly rather than first converting it to $3e^{x} - ke^{2x} - 1 = 0$, which is a quadratic equation in e^{x} (that is, if $u=e^{x}$ then this is equivalent to $-ku^{2}+3u-1=0$).

There were also other methods of solving this problem. One was to observe that the equation $3 - ke^{x} - e^{-x} = 0$ implicitly defines k as a function of x, that is, $k = 3e^{-x} - e^{-2x}$. For $0 < k \leq \frac{9}{4}$, there are one or two values of x that satisfy the equation (by consideration of the graph of this function). Another was to consider the function $f(x) = 3 - ke^{x} - e^{-x}$ and find the coordinates (x, y) of the turning point, and require that the y value of the turning point is non-negative. (The coordinates of the turning point are $\left(-\frac{\log_e(k)}{2}, 3-2\sqrt{k}\right)$).

Question 4

4ai.

Marks	0	1	Average
%	14	86	**0.9**

Since $f(0) = p(0)^3 + q(0)^2 + r(0) + s = 7$, then $s = 7$

4aii.

Marks	0	1	Average
%	22	78	**0.8**

$f'(x) = 3px^2 + 2qx + r$

$f'(0) = 4.25$, so $r = 4.25$

This question was generally well done.

4bi.

Marks	0	1	2	Average
%	50	9	41	**1.0**

$f'(1)=0$ so $3p + 2q + 4.25 = 0$

$$q=\frac{-17-12p}{8}$$

Some students tried to use an expression involving $f(1)$. Some wrote $q=-\frac{17-12p}{8}$, being careless with the two '–' signs.

4bii.

Marks	0	1	Average
%	66	34	**0.4**

$f(1) = p + q + 4.25 + 7$

$= p + \frac{-17-12p}{8} + 4.25 + 7$

$= -\frac{p}{2} + \frac{73}{8}$

4biii.

Marks	0	1	Average
%	73	27	**0.3**

$f'(x) = 3px^2 + 2qx + 4.25$

$= 3px^2 + 2\frac{(-17-12p)}{8}x + 4.25$

$= \frac{(12px-17)(x-1)}{8}$

hence $a = \frac{17}{12p}$

Some students tried to use the quadratic formula rather than factorising, and gave two answers, one of which was 1. They needed to explicitly identify the required result.

4biv.

Marks	0	1	2	Average
%	61	13	26	**0.7**

$\frac{17}{12p} = \frac{17}{3}$, so $p = 0.25$

and $q = \frac{-17-12\times 0.25}{8} = -2.5$

Many students simply substituted the values for p and q into the equation $f'\left(\frac{17}{3}\right) = 0$ and showed that the equation held with these values. Unfortunately there are infinitely many such pairs of values for p and q which satisfy this equation.

4c.

Marks	0	1	2	Average
%	22	26	51	**1.4**

$f(x) = 0.25(x-7)(x-4)(x+1)$, so D has coordinates (4, 0) and F has coordinates (7, 0).

This question was quite well done using CAS, although some students did not present their answers as coordinates.

4d.

Marks	0	1	Average
%	45	55	**0.6**

3.70 units

This was generally well done by the students who attempted it. Some students did not round the answer correctly, especially by dropping off the last zero to give 3.7 as their answer. Some gave the corresponding x-value instead of the distance.

4e.

Marks	0	1	2	3	Average
%	43	18	8	31	**1.3**

Solve $\{g(0) = 7, g'(0) = 4.25\}$ for *a* and *b*.

$a = 7$ and $b = \frac{17}{28}$

Most students who attempted this question were able to obtain the correct value for *a*.

4f.

Marks	0	1	2	3	4	Average
%	48	3	4	18	28	**1.8**

$$\int_{-2}^{0} g(x)dx + \int_{0}^{4} f(x)dx \approx 33.83$$

The area is 33.83 square units, correct to two decimal places.

Most students who attempted this question wrote an expression for the required area as a sum of two definite integrals, and then used technology to evaluate the definite integrals.