NEW DIRECTIONS 2014

1. Introduction

1.1 Motivation

In the past two years there has been a a range of events that have promoted broader discussion about future directions for senior secondary mathematics curriculum in Victoria, including

- The release of the Australian Curriculum in Mathematics for the senior curriculum by the Australian Curriculum, Assessment and Reporting Authority [1] There was extensive consultation across the country with members of the community including mathematicians, teachers, representatives from the state education authorities, and teacher educators. There was also systematic benchmarking against the courses of several other systems of other countries. State and territory curriculum, assessment and certification authorities are responsible for how senior secondary courses are organised, and they will determine how the Australian Curriculum content is to be integrated into their courses.
- *The 21st Century Mathematics* [2] conference run in Stockholm in April 2013. It was organised by the Centre for Curriculum Redesign in collaboration with the OECD and the Confederation of Swedish Enterprise. The discussion centred around the question 'What students should learn in the 21 st century?' The rationale was the following strong statement.

  > *In the 21st century, humanity is facing severe difficulties at the societal (global warming, financial stresses), economic (globalisation, innovation) and personal levels (employability, happiness). Technology's exponential growth is rapidly compounding the problems via automation and off-shoring, which are producing social disruptions. Education is falling behind the curve, as it did during the Industrial Revolution. The last profound changes to curriculum were effected in the late 1800's as a response to the sudden growth in societal and human capital needs. As the world of the 21st century bears little resemblance to that of the 19th century, education curricula are overdue for a major redesign.*

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[1] The senior mathematics curriculum is found at http://www.australiancurriculum.edu.au/SeniorSecondary/Overview

[2] http://eventus.trippus.se/21stcenturymaths

> *This is all the more true in Science/Technology/Engineering/Math (STEM), where demand is outpacing supply worldwide. Math being the foundation of STEM, and in turn innovation, the situation requires urgent attention. Beyond STEM professions, we are seeing very significant innumeracy in a very large segment of the population, which has severe consequences on the ability to understand the world's difficulties.*

- A joint VCAA/DEECD forum in Melbourne in June 2014 with Charles Fadel, the organiser of the Stockholm conference, which included speakers from industry. The forum provided an opportunity to discuss the kind of mathematics curriculum wanted now and into the future, and was convened to examine the extent to which the current Victorian curriculum meets those demands and how it might need to evolve in future years.
- A meeting with Conrad Wolfram in May 2014 convened by the Secretary of the Department, . An aspect of the conversation was the following.

> *What's wrong with today's math?*
>
> *It's 80 % a different subject from what is required.*
>
> *Why?*
>
> *Because computers mechanised computation beyond previous imagination and do calculating really well. Today's math education spends 80 % of the curriculum time gaining expertise in hand-calculation methods and algebraic manipulation. The curriculum is ordered by the difficulty of the skills necessary to complete the calculation, rather than the difficulty of understanding the complexity of the topic.*

- A meeting with the industry sub-committee of the Decadal Plan for Mathematics which is being overseen by the Australian Academy of Science. The meeting listened to four members of the subcommittee and heard of their views. They could see the need for a change in the way mathematics is taught.
- The review of the VCE mathematics study, and preparation of the draft for consultation of the proposed Victorian mathematics study design for years 11 and 12[3].

  This took place 2013-2014,starting with the work of the Mathematics Expert Reference panel convened by the VCAA to provide advice on directions for the review, and the VCAA Proposed directions discussion paper (June 2013). Comprehensive bench marking with similar courses from other countries was undertaken.

[3] http://www.vcaa.vic.edu.au/Documents/vce/mathematics/MathsSD_consultation_draft.pdf

1.2 Purpose

The purpose of this paper is to connect the possible directions for the evolution of the Senior Secondary Mathematics Curriculum in Victoria with the

- different pathways that research in mathematics has taken and the areas that are considered important by eminent mathematicians.
- application of mathematics in a vast number of areas
- development of mathematics in the twentieth century and the present century.
- use of technology in mathematics and its applications
- views of some mathematicians and committees of mathematicians on what has happened in the past century and what they think will be the directions of mathematics and its applications in the future.

A necessity for achieving these connections is to briefly outline the development and history of the senior mathematics curriculum in Victoria

1.3 Structure

This paper has the following sections.

1. Introduction
2. Twentieth Century Mathematics
3. Twenty-first Century Mathematics and Statistics
4. Contemporary Applications
5. Possible implications for mathematics education
6. The Victorian Senior Mathematics Curriculum
7. Concluding remarks

This paper has been developed for a general reader and draws on the views of a range of people who are well regarded in their field, and well placed to comment on to what is important in mathematics.

1.4 The growth of mathematics

It does not seem to be common knowledge that mathematics is both changing and increasing. The twentieth century mathematician Alexander Ostrowski once said that when he came up for his qualifying examination in Germany in 1915 it was expected that he would be prepared to deal with any question in any branch of mathematics. In the late 1940's John von Neumann estimated that a skilled mathematician might know , in essence ten percent of what is known. Today MathSciNet[4], an electronic collection of Reviews of the American Mathematical Society contains information on about 2 million articles from 1,900 mathematical journals. The amount of published mathematics is increasing rapidly. Even the most accomplished mathematician is now is familiar with only a small percentage of the totality of mathematical knowledge. It is clear that the mathematical output of the twentieth century exceeded that of all previous centuries put together. This growth shows no signs of abating.

1.5 What is a mathematician?

Since we are discussing the relationship between developments in mathematics and our curriculum it is worthwhile pausing for a moment to get at a view on what mathematics is. The following is not a philosophical view, but a practical job description that comes from the United States Department of Labour [5]

Mathematicians typically do the following:

- Expand knowledge in mathematical areas, such as algebra or geometry, by developing new rules, theories, and concepts
- Use mathematical formulas and models to prove or disprove theories
- Apply mathematical theories and techniques to solve practical problems in business, engineering, the sciences, or other fields
- Develop mathematical or statistical models to analyse data
- Interpret data and report conclusions from their analyses
- Use data analysis to support and improve business decisions

The following are examples of types of mathematicians again given by the United States Department of Labour:

Applied mathematicians use theories and techniques, such as mathematical modelling, to solve practical problems. These mathematicians typically work with individuals in other occupations to solve these problems. For example, they may work with chemists and materials scientists and chemical engineers to analyse the

[4] http://www.ams.org/mathscinet/

[5] http://www.bls.gov/ooh/math/mathematicians.htm#tab-2

effectiveness of new drugs. Other applied mathematicians may work with industrial designers to study the aerodynamic characteristics of new automobiles.

Theoretical mathematicians do research to identify unexplained issues in mathematics and resolve them. They are primarily concerned with exploring new areas and relationships of mathematical theories to increase knowledge and understanding about the field. Although some may not consider the practical use of their findings, the knowledge they develop can be an important part of many scientific and engineering achievements.

Despite the differences, these areas of mathematics frequently overlap. Many mathematicians will use both applied and theoretical knowledge in their work.

1.6 What is a statistician?

This is again from the United States Department of Labour. [6] Statisticians must develop techniques to overcome problems in data collection and analysis. Statisticians use statistical methods to collect and analyse data and help solve real-world problems in business, engineering, the sciences, or other fields.

Statisticians typically do the following:

- Apply statistical theories and methods to solve practical problems in business, engineering, the sciences, or other fields
- Decide what data are needed to answer specific questions or problems
- Determine methods for finding or collecting data
- Design surveys or experiments or opinion polls to collect data
- Collect data or train others to do so
- Analyse and interpret data
- Report conclusions from their analyses
- Statisticians design surveys, questionnaires, experiments, and opinion polls to collect the data they need. They may also write instructions for other workers on how to collect and arrange the data. Surveys may be mailed, conducted over the phone, collected online, or gathered through some other means.

While distinctive aspects of discourse in mathematics and statistics is sometimes highlighted they are closely related fields.

[6] http://www.bls.gov/ooh/math/statisticians.htm#tab-2

What is mathematics?

It is worth recounting a variety of views about the importance, nature and role of Mathematics Here is a definition of what Mathematics is by Keith Devlin [7]

> *According to this new definition, what the mathematician does is examine abstract patterns-numerical patterns, patterns of shape, patterns of motion, patterns of behaviour, voting patterns in a population, patterns of repeating chance events, and so on. Those patterns can be either real or imagined, visual or mental, static or dynamic, qualitative or quantitative, purely utilitarian or of little more than recreational interest. They can arise from the world around us, from the depths of space and time, or from the inner workings of the human mind. Different kinds of patterns give rise to different branches of mathematics. For example:*
>
> - *Arithmetic and number theory study the patterns of number and counting.*
> - *Geometry studies the patterns of shape.*
> - *Calculus allows us to handle patterns of motion (including issues such as velocity and acceleration, polynomial motion, exponential motion, etc.).*
> - *Logic studies patterns of reasoning.*
> - *Probability theory deals with patterns of chance.*
> - *Topology studies patterns of closeness and position.*
>
> and so forth.

1.7 Up to the twentieth century

This is not the place to even briefly outline history of mathematics but it is worth noting that it has not always been the same. In early civilisations mathematics was about doing arithmetic. Modern mathematics dates back to the ancient Greeks from about 500 B.C. It was with the Greeks that Mathematics became an identifiable discipline.The geometry introduced by the Greeks is still part of our school curriculum and the teaching of it has been strengthened in the AusVELS now being taught in Victorian schools. The next major change was the development of Calculus by Isaac Newton and Gottfied Liebniz in the seventeenth century. Their work is studied by Victorian student in years 11 and 12 at a suitable level. Gradually from about the middle of the eighteenth

[7] Devlin, K. (2008). What will count as mathematics in 2100? In Proof and other Dilemmas (Gold, B., Simons, R. A., eds.) pp291-311. The Mathematical Association of America.

century the old idea of formal proof from the ancient Greeks became important. In the nineteenth century topics such as probability and logic began to be studied. Both of these topics are considered at an introductory level in the Senior Victorian Curriculum.

We now quote directly from Keith Devlin [8]

> *In the middle of the 19th century, however, a revolution took place. Generally regarded as having its epicenter in the small university town of Göttingen in Germany, the revolution's leaders were the mathematicians Lejeune Dirichlet, Richard Dedekind, and Bernhard Riemann. In their new conception of the subject, the primary focus was not performing a calculation or computing an answer, but formulating and understanding abstract concepts and relationships. This represented a shift in emphasis from doing to understanding. For the Göttingen revolutionaries, mathematics was about 'Thinking in concepts' (Denken in Begriffen). Mathematical objects, which had been thought of as given primarily by formulas, came to be viewed rather as carriers of conceptual properties. Proving was no longer a matter of transforming terms in accordance with rules, but a process of logical deduction from concepts.*

It was at this stage the concept of function was introduced. The definition of function given by Dirichlet (1805 -1859) is still the one introduced in Victorian schools today. It is interesting that this has been a central concept in the Victorian senior curriculum since the early 1970s. This will be discussed further in the section below. It is a central concept of our curriculum.

With the concepts introduced at this time mathematicians had a a rigorous way of handling infinity, a concept that their predecessors had all struggled with.

Devlin believes that the next revolution will leave the nature on mathematics unchanged but will look very different on the surface. The reason for this is that it will be applied to areas which have a significant degree of non-determinism or such complexity that it defies capture by a traditional mathematical framework intelligible to us.

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[8] Devlin, K. (2008). What will count as mathematics in 2100? In Proof and other Dilemmas (Gold, B., Simons, R. A., eds.) pp291-311. The Mathematical Association of America.

1.8 Why mathematics?

Here we quote Professor Brian Schmidt (Nobel prize winner (Physics),2011) [9] who spoke eloquently and powerfully of the importance of mathematics at the AMSI "Maths for the future: Keep Australia competitive' forum, held from 7-8 February 2012 at ANU. This is followed by a quote from Keith Devlin at the Stockholm conference with a view about what mathematically trained people can offer to the community.

> *Everyone in Australia - and I mean everyone - needs to be mathematically literate, or numerate as we like to say, and our country needs many people to be more than numerate: we need people to be highly skilled.*
>
> *For me, the tools of mathematics go hand in hand with the astronomy I undertake. Each day I spend more time using mathematics than any other activity. I took eight classes at university in mathematics, almost as many classes as I did in physics, and twice as many as I took in astronomy.*
>
> *Now, you may be thinking that I am special - but let's just look at my family. My father is a biologist, studying the populations of fish stocks in Alaska and now Canada - he uses sophisticated mathematics every day to understand exactly how to ensure that fish stocks remain at healthy levels into the future - as people fish, or dams release water, or glacial run-off slows or speeds up. Ah, but he is a scientist, you say. True.*
>
> *My wife is an economist - whom I met at Harvard. Her education in economics has almost as much math in it as mine. Solving challenging coupled differential equations, undertaking sophisticated statistical tests - all to ensure that economies work efficiently at allowing their people to be prosperous - it is way more than simply bean counting.*
>
> *But we all have PhDs. My Australian cousin and her husband who work in the mining industry as engineers - maths is the fundamental basis of their work, and, for that matter, Australia's ability to extract minerals and become one of the world's most wealthy countries.*
>
> *My other cousin and his wife are farmers in Western Australia who do precision farming, where fertilisers and seeds are linked to a GPS system, and planted out at optimum values - all calculated by them using, you guessed it - math. Farming runs on tiny margins - a few percent - and this sophistication allows them to make money when others go bankrupt.*

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[9] Speech: Brian Schmidt's mathematical argument, Brian Scmidt, The Australian, February 09, 2012

Keith Devlin a highly respected mathematician and popularist said the following at the Stockholm conference.

> *I have a bachelors and a doctorate in mathematics.*
>
> - *Every technique and method I learned at university can now be outsourced to where it can be completed faster and more cheaply.*
> - *The only things I have left of market value are experience, a human network, and a powerful collection of metacognitive skills including mathematical thinking.'*

Keith Devlin 2012

2. Twentieth Century Mathematics and Statistics

It is impossible to quickly summarise twentieth century mathematics and statistics and so my choice has been to follow paths which have either changed school mathematics or evidently may influence what happens in the future. The first half of the century is easier because the work and directions of David Hilbert and then the Bourbaki group clearly had a world wide effect on school education. Some of the aspects of this influence have been viewed in a negative way but in Victoria we still feel the effect of Hilbert and Bourbaki in our school curriculum.

We also briefly refer to the work of Felix Klein. This was one of the first times an eminent mathematician was seen to influence a contemporary educational system - in this case in Germany at the turn of the twentieth century. Today there is project run by the International Mathematical Union which endeavours to replicate Klein's work in the present age.

Two results, Fermat's last theorem and the four colour theorem are then briefly discussed. The four colour theorem was chosen as it was the first time that computer results were used in the proving of a theorem. Fermat's last theorem stands out as a result that was claimed by Fermat in the border of a greek mathematical text in the eighteenth, but the proof of which eluded some of the greatest mathematicians of the intervening centuries and was proved finally at the end of the twentieth century. It is an example of a very hard problem which is not very important in itself but in the search for its solution by many mathematicians a great deal of important mathematics has been written.

As an application in a different field we refer to the Nobel Prize in Economics for 1997. The winners were Professor Robert C. Merton, of Harvard University, Cambridge, USA and Professor Myron S. Scholes, of Stanford University, Stanford, USA, for the discovery of 'a new method to determine the value of derivatives' .

The traditional connection between physics and mathematics was refreshed in the last decades of the twentieth century. The relationship was beneficial to both disciplines. We give the account of this by Sir Michael Atiyah.

It is impossible to include all of the important achievements in mathematics in the twentieth century. For example, the work of Alan Turing and Claude Shannon that lead us towards the computing that we see today has not been included.

There are endless applications of mathematics in the world today. We mention a few of these in section 4. Contemporary applications.

2.1 Hilbert

David Hilbert was born in Königsberg, East Prussia in 1862 and received his doctorate from his home town university in 1885. His knowledge of mathematics was broad and he excelled in most areas.

In 1900, when he was 38 years old, Hilbert gave a massive homework assignment to all the mathematicians of the world. This was done when he presented a lecture, entitled '*Mathematical Problems*' before the International Congress of Mathematicians in Paris.

In the biography of Hilbert by Constance Reid [10] it is written that if any mathematician can be said to be a history of mathematics in his time, it is that of David Hilbert. His remarkable prescient proposal in 1900 problems for the forthcoming century set the course of much consequent mathematics. When he died in 1943, It was remarked in *Nature* that there was scarcely a mathematician in the world whose work didn't derive from Hilbert.

Hilbert began his Paris address with the following.

> *Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries ? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive ? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose ? History teaches the continuity of the development of science. We know that every age has its own*

[10] Constance Reid, Hilbert, Springer-Verlag 1996

problems, which the following age either solves or casts aside as profitless and replaces by new ones. If we would obtain an idea of the probable development of mathematical knowledge in the immediate future, we must let the unsettled questions pass before our minds and look over the problems which the science of to-day sets and whose solution we expect from the future. To such a review of problems the present day, lying at the meeting of the centuries, seems to me well adapted. For the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future.

In 1950, when Hermann Weyl (another great mathematician of the twentieth century) was asked by the American Mathematical Society to summarise the history of mathematics during the first half of the twentieth century, he wrote that if the terminology of the Paris Problems had not been so technical he could have performed the required task simply in terms of Hilbert's problems which had been solved or partially solved - 'a chart by which we mathematicians have often measured our progress' during the past 50 years.

In 1962 Richard Courant said the following at a meeting in Göttingen on the hundredth anniversary of Hilbert's birth.

. . . I feel that the consciousness of Hilbert's spirit is of great actual importance for mathematics and mathematicians today. . . . I believe that we find ourselves in a period of danger. In our time of mass media, the call for reform, as a result of propaganda, can just as easily lead to a narrowing and choking as to a liberating of mathematical knowledge. That applies, not only to research in the universities but also instruction in schools. . . .

Living mathematics rests on the fluctuation between the arithmetical powers of intuition and logic, the individuality of 'grounded' problems and the generality of far-reaching abstractions.

2.2 Bourbaki

In the early 1930's mathematics in France was still recovering from the first world war. Little influence from what was happening elsewhere in Europe, in particular the flourishing German groups. Some young French mathematicians were visiting these groups and wanted to change things in France. A group of young mathematicians decided to write together to represent the essence of contemporary mathematics. They were unhappy with most of the existing texts and wanted to adopt a more precise rigorous style of exposition than had been the case in the past in France.

They chose to write under the pen name Nicholas Bourbaki , who was a French general of the First World War. The first book appeared in 1939. The initial books were to be called as a group Éléments de mathématique ('Elements of Mathematics') [11].

The first book was on set theory. In the 1940's several more books were completed. There is no doubt that there work was highly influential through the middle decades of the twentieth century. I was not just the one group of mathematicians and many of the great French mathematicians of the twentieth century have been involved. Up to this point there are over 30 volumes produced.

Sir Michael Atiyah in his book review of 'A Secret Society of Mathematicians and The Artist and the Mathematician'[12],writes

> *So what were the basic aims of Bourbaki, and how much was achieved? Perhaps one can pick out two central objectives. One was that mathematics needed new and broad foundations, embodied in a series of books that would replace the old- fashioned textbooks. The other was that the key idea of the new foundations lay in the notion of 'structure', illustrated by the now common word 'isomorphism'.*
>
> *There is no doubt that, with its clear emphasis on 'structure', Bourbaki produced the right idea at the right time and changed the way most of us thought. Of course it fitted in well with Hilbert's approach to mathematics and the subsequent development of abstract algebra. But structure was not confined to algebra, and it was particularly fruitful in topology and*

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[11] Éléments de mathématique Théorie des ensembles, chapitres 1 à 4, 352 p., 1970 (réimpression en 1998); Algèbre, chapitres 1 à 3, 654 p., 1970; Algèbre, chapitres 4 à 7, 432 p., 1981; Algèbre, chapitres 8, 489 p., 2012; Algèbre, chapitre 9, 212 p., 1959 (nouveau tirage en 1973); Algèbre, chapitre 10, 224 p., 1980; Topologie Générale, chapitres 1 à 4, 376 p., 1971 (réimpression en 1990); Topologie Générale, chapitres 5 à 10, 334 p., 1974; Fonctions d'une Variable Réelle, chapitres 1 à 7, 336 p., 1976 Espaces Vectoriels Topologiques, chapitres 1 à5, 400 p., 1981; Intégration, chapitres 1 à 4, 284 p., 1965 (nouveau tirage en 1973); Intégration, chapitre 5, 154 p., 1967; Intégration, chapitre 6, 106 p., 1959; Intégration, chapitres 7 et 8, 222 p., 1963; Intégration, chapitre 9, 134 p., 1969; Algèbre commutative, chapitres 1 à 4, 364 p., 1968 et 1969 (réimpression en 1985); Algèbre commutative, chapitre 5 à 7, 352 p., 1964et 1965 (réimpression en 1985); Algèbre commutative, chapitres 8 et 9, 208 p., 1983; Algèbre commutative, chapitre 10, 187 p., 1998; Variétés différentielles et analytiques (fascicule de résultats), 198 p., 1971 (réimpression en 1998); Groupes et algèbres de Lie, chapitre 1, 146 p., 1971; Groupes et algèbres de Lie, chapitres 2 et 3, 320 p., 1972; Groupes et algèbres de Lie, chapitres 4 Ãă 6, 288 p., 1968 (réimpression en 1981); Groupes et algèbres de Lie, chapitres 7 et 8, 272 p., 1975 (réimpression en 1998); Groupes et algèbres de Lie, chapitre 9, 144 p., 1982; Théories spectrales, chapitres 1 et 2, 168 p., 1967; Éléments d'histoire des mathématiques, 376 p., 1974 (réimpression en 1984)

[12] Bourbaki, A Secret Society of Mathematicians and The Artist and the Mathematician Reviewed by Michael Atiyah, Notices of the AMS Volume 54, Number 9,October 2007

> *associated areas of geometry, all of which were to see spectacular developments in the period following World War II.*

There is no doubt that Bourbaki influenced the Mathematics that was taught in schools and the education reforms in Victoria of the period 1967 -1971 can be directly linked to the development and influence of Bourbaki.

On this Sir Michael Atiyah [13] writes

. . . much of the critique directed against Bourbaki is that it was used, or perhaps misused, to reform school education. This may be unfair, since many of the great mathematicians in Bourbaki were excellent lecturers and knew well the difference between formal exposition and the conveying of ideas. But, as so often happens, the disciples are more extreme and fanatical than their masters, and education in France and elsewhere suffered from a dogmatic and ill-informed attempt at reform.

Amand Borel[14] was a member of the Bourbaki writing group. He finishes his article ... with

> *Of course, Bourbaki has not realised all its dreams or reached all of its goals by far. Enough was carried out, it seems to me, to have a lasting impact on mathematics by fostering a global vision of mathematics and of its basic unity and also by the style of exposition and choice of notation, but as an interested party I am not the one to express a judgment.*
>
> *What remains most vividly in my mind is the unselfish collaboration over many years of mathematicians with diverse personalities toward a common goal, a truly unique experience, maybe a unique occurrence in the history of mathematics. The underlying commitment and obligations were assumed as a matter of course, not even talked about, a fact which seems to me more and more astonishing, almost unreal, as these events recede into the past.*

2.3 Felix Klein 1849 -1925 and the Klein project

Before leaving the first half of the twentieth century it is appropriate to mention another German mathematician, Felix Klein. He was interested in the education of teachers of

13 Bourbaki, A Secret Society of Mathematicians and The Artist and the Mathematician Reviewed by Michael Atiyah,Notices of the AMS Volume 54, Number 9,October 2007

14 Amand Borel, Twenty-Five Years with Nicolas Bourbaki,Notices of the AMS Volume 45, Number3,March 1998

mathematics and the manner in which mathematics was taught in the schools. He wrote a three-volume work, '*Elementary Mathematics from an Advanced Standpoint*' [15], whose goal was to help teachers to bridge the gap between the subject matter taught at university and the usual topics that were covered in the schools and to integrate them into the curriculum. He was vehement in his belief in the need to strengthen those aspects of instruction that gave students the capacity to think in three dimensions.

He promoted education in functional thinking and influenced the treatment of differential and integral calculus as a unified topic of instruction in secondary school Klein was the head of the International Commission on Mathematical Education and this took his influence beyond the borders of Germany. His work also indirectly influenced what happened in Australia. A review by G B Price [16] is worth including to show the influence of these books.

> *Klein possessed in an unusual degree the abilities of a great mathematician and the gifts of an inspiring teacher and lecturer. He had a broad knowledge of mathematics and a correspondingly deep insight into the foundations and interrelations of its various branches. Both Klein's qualifications for writing a book of this nature and the scarcity of such books combine in directing attention to the present volume.*
>
> *This book, a translation of the first of Klein's three volumes entitled Elementarmathematik vom höheren Standpunkte aus, is a series of lectures that Klein gave for teachers of mathematics in secondary schools. The material is presented under the headings of arithmetic, algebra, analysis, and a supplement. The section on arithmetic treats the extensions of the number system and the laws of operation, beginning with integers and ending with complex numbers and quaternions. The treatment seeks to explain the how and why of the subject. As an example, we note the discussion of the little understood rule of signs: 'minus times minus gives plus.' The section on algebra is devoted to the solution of equations. First, some geometric methods are explained for investigating the real roots of rational integral equations containing parameters. Then complex roots are considered, especially of those equations whose solutions lead to a consideration of the groups of motions connected with the regular bodies. Free use is made of Riemann surfaces and other parts of the theory of functions of a complex*

[15] Elementary Mathematics from an Advanced Standpoint. Arithmetic, Algebra, Analysis. By Felix Klein. Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York, Macmillan, 1932.

[16] Bull. Amer. Math. Soc. Volume 39, Number 7 (1933), 495-496. Review: Felix Klein, Elementary Mathematics from an Advanced Standpoint. Arithmetic, Algebra, Analysis

> *variable. The section on analysis is devoted to the logarithmic, exponential, and trigonometric functions, and a discussion of the infinitesimal calculus proper. A wide variety of subjects is treated, however, in connection with these general topics: the construction of the early logarithmic and trigonometric tables, expansions in Fourier series, Taylor's Theorem, and Newton's and Lagrange's interpolation formulas will serve as samples.*
>
> *The real excellence of the book, however, is due to certain clearly defined characteristics of the presentation. In the first place, the historical development of the theory is traced. This is not history for history's sake alone, but history as an aid to gaining a deeper insight into the present state of the theory. In this connection it should be stated that the inductive method of presentation is used exclusively.*
>
> *Secondly, the geometric aspects of the subjects treated are emphasised. It is significant that the book contains 125 figures. The geometric meaning of Fermat's Theorem is explained; the Pythagorean number triples are obtained by a geometric method. The graphs of the approximating polynomials of Taylor's series expansions are drawn in order to show the nature of the convergence and divergence; similarly for Fourier series. Klein would develop geometric intuition and sense perception as an aid to mathematical investigation.*
>
> *Again, Klein shows the mutual relations between problems in different fields. His ability to discover such relations is well known, and many examples are to be found in this volume.*

The Klein Project [17] which began in 2009 is a major project of the International Mathematical Union (IMU) and the International Commission on Mathematical Instruction (ICMI). It aims to link research mathematics with school mathematics in the spirit of Felix Klein's books that were published in 1908. Since then mathematics has grown exponentially, changed under the influence of computers and new fields and applications, adopted new approaches, and developed new concepts. The project aims to consider the following questions:

- How can fundamental contemporary ideas in research mathematics get related to the mathematical foundation required in today's world?
- How has research mathematics been guided by contemporary social interests, how might that happen in the future, and is this a good thing?

[17] Klein Project, http://www.projekt.didaktik.mathematik.uni-wuerzburg.de/klein/index.html

- What influence do contemporary trends in the discipline of mathematics have on current secondary curricula, and what influence should it have?
- How can teachers' mathematical knowledge be kept up to date in a world of accelerating growth in the mathematical sciences?
- What forms of communication between researchers and teachers are optimal?

These are indeed challenging aims.

2.4 Foundations

Generally people make statements like I like mathematics because you know where you stand – you are either right or wrong or that mathematics can provide truth and certainty. This subsection is included to give the idea that not all is as straightforward in mathematics as thought by all. I rely on the excellent exposition by Davis and Hersch in '*The Mathematical Experience*'[18].

According to naive set theory, any definable collection is a set. Let ℝ be the set of all sets that are not members of themselves. If ℝ is not a member of itself, then its definition dictates that it must contain itself, and if it contains itself, then it contradicts its own definition as the set of all sets that are not members of themselves. This contradiction is Russell's paradox. Another version of it is. 'A barber in a certain town has stated that he will cut the hair of all those those persons and only those persons in the town who do not cut their own hair. Does the barber cut his own hair?'

When Russell communicate this it was considered to be a crisis in foundations, and efforts were made to reformulate set theory so that Russell's paradox could be avoided. The work on this program played a major role in the development of logic but it failed in its aim in find a suitable framework. There was no resolution and when Gödel proved his incompleteness theorems in 1930 it was believed by most that any consistent formal system strong enough to contain elementary arithmetic would be unable to prove its own consistency. The search for secure foundations never recovered from this defeat.

There was further work on this type of result. In 1963 Paul Cohen solved one of Hilbert's problems but in an unexpected way by showing that within the confines of a commonly used set of axioms for set theory the result could neither be proved or disproved. With these and more recent results it seems to be in set theory that there is not one version but many possible versions depending on your set of axioms.

In any discussion of the foundations of mathematics three standard dogmas are

[18] Davis PJ, Hersh, R, Marchisoto EA, The Mathematical Experience, Study Edition, Birkhauser, 199.

presented: Platonism, formalism and constructivism.

- Platonism is the metaphysical view that there are abstract mathematical objects whose existence is independent of us and our language, thought, and practices. Just as electrons and planets exist independently of us, so do numbers and sets. And just as statements about electrons and planets are made true or false by the objects with which they are concerned and these objects' perfectly objective properties, so are statements about numbers and sets. Mathematical truths are therefore discovered, not invented.
- Formalism is the view that there are no mathematical objects. Mathematics just consists of axioms, definitions and theorems. Mathematics is not a body of propositions representing an abstract sector of reality but is much more akin to a game. Of course the formalist knows that mathematical formulas are sometimes applied to physical problems and if given a physical meaning it acquires a meaning and may be true or false.

Formalists and Platonists are at opposite sides on the question of existence and reality but they have no quarrel on what principle of reasoning should be. The actual reality for working mathematicians is that they switch between these two views on what mathematics is.

- Mathematical constructivism asserts that it is necessary to find or construct a mathematical object to prove that it exists. We will not go into this here. There are different forms of constructivism. The reality is that it is sometimes at odds with the other two beliefs.

John Dieudonne said the following

> *We believe in the reality of mathematic but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say , 'Mathematics is just a combination of meaningless symbols', and then we bring out chapters 1&2 of set theory. Finally we are left in peace to go back to our mathematics ad do as we have always done, with the feeling each mathematician has that he is working with something real. This sensation is probably an illusion, but is very convenient. That is Bourbakis' attitude to foundations.*

In *The Mathematical Experience* the following observation is made regarding the effect on education in schools: The aim for axiomatisation of mathematics and the formalist view led to the unfortunate importation into primary and secondary schools during the 1960's, of set theoretic notation and axiomatics. It was a predictable consequence of a philosophical doctrine: Mathematics is axiomatic systems expressed in set-theoretic language.

Critics of formalism in high school say 'This is the wrong thing to teach and the wrong way to teach' Such criticism leaves unchallenged the dogma that real mathematics is formal derivations from formally stated axioms. If this dogma rules, the critic of formalism is seen as asking for lower quality. The fundamental question is, 'What is mathematics?' Controversy about high school teaching can't be resolved without controversy about mathematics.

2.5 Fermat's last theorem [19]

Fermat's Last Theorem states that no three positive integers a, b, and c can satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than two. If $n = 2$ it is the Pythagorean equation $a^2 + b^2 = c^2$ which has infinitely many solutions This theorem was conjectured by Pierre de Fermat in 1637, in the margin of a copy of the ancient Greek book, Arithmetica, where he claimed he had a proof that was too large to fit in the margin.

There was some progress up to the end of the twentieth century where initially the theorem was proved in the cases for $n = 3$ and $n = 4$. By 1993 it had been proved to be valid for all for all exponents up to 4×10^6 . No counterexamples had been found and the evidence pointed towards the theorem being true. In 1993, the general theorem was partially proven by Andrew Wiles. Unfortunately, several holes were discovered in the proof . However, the difficulty was circumvented by Wiles and R. Taylor in late 1994 and published in 1995

The proof of Fermat's Last Theorem marks the end of a mathematical era. Since virtually all of the tools which were eventually brought to bear on the problem had yet to be invented in the time of Fermat, it is interesting to speculate about whether he actually was in possession of an elementary proof of the theorem. Judging by the tenacity with which the problem resisted attack for so long this is not likely.

The method used to obtain the solution is of far more importance to mathematics than the Last Theorem itself.

Wiles result, and the work of the many other mathematicians that paved the way, is sure to have enormous impact in many parts of mathematics.

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[19] There are many possible references but the following is a succinct readable article, A History of Mathematics: An Introduction.Victor J. Katz, Addison Wesley Publishing Company Incorporated, 2009

2.6 The four-colour theorem[20]

The four-colour theorem states that any map in a plane can be coloured using four colours in such a way that regions sharing a common boundary (other than a single point) do not share the same colour. The problem was first put forward in 1852. Several proofs containing errors were presented over the next century and one of these proofs was even accepted for ten years.

This result was finally obtained by Appel and Haken , who constructed a computer-assisted proof that four colours were sufficient. However, because part of the proof consisted of an exhaustive analysis of many discrete cases by a computer and it is not accepted by some mathematicians. However, no mistakes have been found and so the proof appears valid. A shorter, independent proof was constructed, again using a computer, and this has been verified. The Four Colour Theorem was the first major theorem to be proved using a computer, having a proof that could not be verified directly by other mathematicians. Despite some worries about this initially, independent verification soon convinced everyone that the Four Colour Theorem had finally been proved.

A *mathematical assistant* was used to verify the proof. A mathematical assistant is a computer program that a mathematician can use in an interactive way. The mathematician provides ideas and proof steps and the computer carrying out the computations and verification. Such systems have been under development over the last thirty years. Other applications include checking the correctness of computer hardware and software. With this development, it seems that computers have also become indispensable for checking proofs! Mathematics will never be the same again.

An extensive discussion of the four-colour problem is in Keith Devlin's book mentioned in the footnote below.

2.7 An application in economics [21]

On October 14 the Royal Swedish Academy of Sciences announced the winners of the 1997 Nobel Prize in Economics. The winners were Professor Robert C. Merton, of Harvard University, Cambridge, USA and Professor Myron S. Scholes, of Stanford University, Stanford, USA, for the discovery of "a new method to determine the value of

[20] Mathematics: The New Golden Age, Keith Devlin,Penguin 1988.

[21] Ferreyra, Guillernio. 'The Mathematics Behind the 1997 Nobel Prize in Economics.' WhatâĂŹs New in Mathematics 1 January 1998. 29 October 2001 <http://www.ams.org/new-in-math/black-scholesito. html>.

derivatives". The news media had ample coverage of this announcement and of the reason for the award to Merton and Scholes. In the words of the Swedish Academy

Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society. Financial analysts have reached the point where they are able to calculate, with high accuracy, the value of a stock option or derivative. The models and techniques employed by today's analysts are rooted in a model developed by Black and Scholes in 1973, which today is known as the Black-Scholes formula.

What was little discussed in the media at the time the award was announced is the fact that the methodology employed by Merton, Black and Scholes is heavily indebted to the modern mathematical theory of probability. The work of the three economists in the 70's was a novel and extremely useful application to finance of the deep mathematical theory of stochastic processes that had culminated with the theory of stochastic differential equations (SDE's) less than thirty years before.

The theory of SDE's is thriving today due to the many applications it has found in science and engineering, from physics, genetics, and the environmental sciences to electrical engineering and computer science, especially in the de-noising of transmitted data.

The work in economics is just one application of this idea which was first introduced by a Japanese mathematician in the 1940's.

Physics and mathematics [22]

As a consequence of the approach to mathematics as taken by Bourbaki and indeed a lot of mathematicians, pure mathematicians drifted away from applications and saw no need to collaborate with other scientists. Also the application of the highly abstract modern mathematics could not be easily visualised by the traditional users of mathematics. The period from the 1930's to 1970's saw a divergence between mathematics and other applied sciences. Mathematics became more inward looking, and the distinction between pure and applied mathematics became much more pronounced. This situation reversed and often the mathematics being developed had interaction with other disciplines.

[22] Special article: Mathematics in the 20th century, Sir Michael Atiyah, Bull. London Math. Soc. 34 (2002), London Mathematical Society

The following is directly taken from the cited article of Sir Michael Atiyah. It contains quite a lot of reference to specific mathematical structures and the associated physics. However, it does give the reader some idea of the extent of this interaction with physics in the last thirty years of the twentieth century.

> *Throughout history, physics has had a long association with mathematics, and large parts of mathematics, calculus, for example, were developed in order to solve problems in physics. In the middle of the 20th century, this had perhaps become less evident, with most of pure mathematics progressing very well independent of physics, but in the last quarter of this century things have changed dramatically. Let me try to review briefly the interaction of physics with mathematics, and in particular with geometry.*
>
> *In the 19th century, Hamilton developed classical mechanics, introducing what is now called the Hamiltonian formalism. Classical mechanics has led to what we call 'symplectic geometry'. It is a branch of geometry that could have been studied much earlier, but in fact has not been studied seriously until the last two decades. It turns out to be a very rich part of geometry. Geometry, in the sense I am using the word here, has three branches: Riemannian geometry, complex geometry and symplectic geometry, corresponding to the three types of Lie groups. Symplectic geometry is the most recent of these, and in some ways possibly the most interesting, and certainly one with extremely close relations to physics, because of its historical origins in connection with Hamiltonian mechanics and more recently with quantum mechanics.*
>
> *Now, Maxwell's equations, which I mentioned before, the fundamental linear equations of electromagnetism, were the motivation for Hodge's work on harmonic forms, and the application to algebraic geometry. This turned out to be an enormously fruitful theory, which has underpinned much of the work in geometry since the 1930s.*
>
> *I have already mentioned general relativity and Einstein's work. Quantum mechanics, of course, provided an enormous input, not only in the commutation relations but more significantly in the emphasis on Hilbert space and spectral theory. In a more concrete and obvious way, crystallography in its classical form was concerned with the symmetries of crystal structures.*
>
> *The finite symmetry groups that can take place around points were studied in the first instance because of their applications to crystallography. In this century, the deeper applications of group theory have turned out to have*

relations to physics. The elementary particles of which matter is supposed to be built appear to have hidden symmetries at the very smallest level, where there are some Lie groups lurking around, which you cannot see, but the symmetries of these become manifest when you study the actual behaviour of the particles. So you postulate a model in which symmetry is an essential ingredient, and the different theories which are now prevalent have certain basic Lie groups like SU(2) and SU(3) built into them as primordial symmetry groups. So these Lie groups appear as building blocks of matter. Nor are compact Lie groups the only ones that appear. Certain non-compact Lie groups appear in physics, like the Lorentz group. It was physicists who first started the study of the representation theory of non-compact Lie groups. ...

In the last quarter of the 20th century, the one we have just been finishing, there has been a tremendous incursion of new ideas from physics into mathematics. This is perhaps one of the most remarkable stories of the whole century. ...

Physicists have been able to predict that certain things will be true in mathematics based on their understanding of the physical theory. Of course, that is not a rigorous proof, but it is backed by a very powerful amount of intuition, special cases, and analogies. These results predicted by the physicists have time and again been checked by the mathematicians and found to be fundamentally correct, even though it is quite hard to produce proofs, and many of them have not yet been fully proved. So there has been a tremendous input over the last 25 years in this direction. The results are extremely detailed. It is not just that the physicists said, 'This is the sort of thing that should be true.' They said, 'Here is the precise formula and here are the first ten cases (involving numbers with more than 12 digits).' They give you exact answers to complicated problems, not the kind of thing you can guess; things you need to have machinery to calculate. Quantum field theory has provided a remarkable tool, which is very difficult to understand mathematically but has had an unexpected bonus in terms of applications.

This has really been the exciting story of the last 25 years.

2.8 Summary

The changes in secondary mathematics education in the latter part of the twentieth century in someways move in alignment with the developments in mathematics and statistics. We started with teaching calculus together with mechanics and the understanding of the relationship between mathematics and physics (See section 6). This is pedagogically sound but of course there is the drawback that mechanics is not popular with every section of the population and therefore could exclude some people. Anyone who has taught differential calculus at school knows that it is a opportunity to draw together ideas in geometry, algebra and motion in a straight line and can see students responding to the different aspects.The work of Klein and others could have possibly influenced these decisions.

There isn't a doubt that the thoughts of Hilbert and Bourbaki influenced the changes in Victoria in the 1970's. There were some very positive aspects of those changes and because of this these changes remain today. This is also discussed in section 6. The tension between Platonism and formalism is also relevant in thinking about these times.

The growing importance of probability and statistics was recognised and these were introduced into our curriculum and an expansion of these areas in our curriculum is being considered at present.

The move into problem solving and modelling in the 1990's and onto today is paralled by the interaction between different fields of mathematics and the interaction between mathematics and other disciplines.

It would neglectful if we didn't mention the book of the mathematician George Polya 'How to solve it'[23]. It first appeared in 1945 and its influence grew steadily in the second half of the twentieth century and certainly influenced what happened in Victoria from the 1990's.

Sir Michael Atiyah [24] gave the following summary of mathematics in the twentieth century:

> *The 18th and 19th centuries together, were the era of what you might call classical mathematics, the era we associate with Euler and Gauss, where all the great classical mathematics was worked out and developed. You might have thought that would almost be the end of mathematics, but the 20th century has, on the contrary, been very productive indeed. The 20th century can be divided roughly into two halves. I would think the first half was dominated by what I call the 'era of specialisation', the era in which Hilbert's approach, of trying to formalise things and define them carefully and then follow through on what you can do in each field, was very influential. Bourbaki's name is associated with this trend, where people focused*

[23] How to Solve It: A New Aspect of Mathematical Method G. Polya With a foreword by John Conway One of Princeton University Press's Notable Centenary Titles,2014

[24] Special article: Mathematics in the 20th century, Sir Michael Atiyah, Bull. London Math. Soc. 34 (2002), London Mathematical Society

attention on what you could get within particular algebraic or other systems at a given time.

The second half of the 20th century has been much more what I would call the 'era of unification', where borders are crossed over, techniques have been moved from one field into the other, and things have become hybridised to an enormous extent.

2.8 Statistics in the twentieth century [25][26]

The late part of the nineteenth century saw the bringing together of some statistical ideas in England. The measurements that generated the concepts were those of heredity and biometrics. The key statistical ideas of correlation and regression were developed at this time.The chi-squared test was developed by Karl Pearson (1900). This was a tremendously important piece of work,and is still being used extensively.The Department of Applied Statistics at University College in London was founded in 1911 by Karl Pearson, and was the first university statistics department in the world.

R. A. Fisher, also of England, created the foundations of much of modern statistics including the beginning of population genetics. He established methods for the analysis of complex experiments, now called 'analysis of variance', which are used thousands of times each day by scientists around the globe. He showed that a function he called the likelihood could be used to develop optimal estimation and testing procedures in almost any probability model. He founded and developed the main ideas in the design of experiments.

Fisher had a tremendous statistical intuition. A lot of the work in statistics in the twentieth followed on from his work.

Work by Pearson and others developed the theory of hypothesis testing and this became the foundation of research in this area for the remainder of the twentieth century.

Other important advances of the past century came in the area of modelling and estimation, where methods were developed that expanded the horizon of possible models and widened the range of validity of statistical procedures.

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[25] A Report on the Future of Statistics Author(s): Bruce G. Lindsay, Jon Kettenring, David O. Siegmund Source: Statistical Science, Vol. 19, No. 3, (Aug., 2004), pp. 387-407 Published by: Institute of Mathematical Statistics

[26] A History of Mathematics: An Introduction.Victor J. Katz, Addison Wesley Publishing Company Incorporated, 2009

There was also expansion of large sample theory, the study of the distributional properties of statistical procedures when the sample sizes are large. Accurate measures of uncertainty are the key components of statistical inference and large sample methods have enabled statisticians to calculate excellent approximations to these measures in a very wide range of problems.

Beginning in the 1970's, a major revolution in science occurred; it was destined to change the face of statistics forever. The computer has changed completely what it means to carry out a statistical analysis. It has also changed the facility with which one can collect and store data.

3. Twenty First Century Mathematics

This section is divided into three subsections.

- 3.1 The Millenium problems. Seven problems were chosen by the Clay Institute to mark the beginning of the twentieth century
- 3.2 Workshop on directions. A group of eminent mathematicians came together in 2012 to discuss what had happened of importance in the previous decade and what is going to be important in the future.
- 3.3 Computing and pure mathematics

3.1 The Millenium Problems[27]

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) established seven Prize Problems[28]

The prizes were announced at a meeting in Paris, held on May 24, 2000 at the Collège de France. Three lectures were presented: Timothy Gowers spoke on The Importance of Mathematics; Michael Atiyah and John Tate spoke on the problems themselves.

It is of note that one of the seven Millennium Prize Problems, the Riemann hypothesis, formulated in 1859, also appears in the list of twenty-three problems discussed in the

[27] http://www.claymath.org/millennium-problems

[28] The seven problems are: Yang-Mills and Mass Gap Experiment, Riemann Hypothesis, P vs NP problem, Navier-Stokes Equation, Hodge conjecture, Poincaré conjecture and the Birch and Swinnerton-Dyer conjecture

address given in Paris by David Hilbert on August 9, 1900. We quote from a review of Keith Devlin's book which appeared in the London Review of books. [29]

> ### *The Reimann hypothesis*
>
> *Formulated in 1859 by Bernhard Riemann, the hypothesis concerns the distribution of primes among the positive integers. (A prime is any positive integer, other than 1, that is divisible only by itself and 1. The first ten primes are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29.) There are infinitely many primes, but they get rarer and rarer as the sequence of positive integers extends further and further. Thus, of the first eight positive integers, half are primes, but of the first hundred only a quarter are, and of the first million only about one in thirteen are. This raises the question of whether there is anything significant to be said about the precise way in which the proportion gradually decreases. Both the early pattern of primes and what we know about later patterns are discouraging. For instance, the gaps between the first ten primes are 1, 2, 2, 4, 2, 4, 2, 4 and 6, a series which does not exhibit any obvious regularity. Furthermore, no matter how far out along the [positive integers] you go, you can find clusters of several primes close together as well as stretches as long as you like in which there are no primes at all. The distribution of primes is the part of mathematics with the greatest feel of contingency about it: primes seem to crop up at random, like rocks scattered on a barren landscape. Yet mathematicians have achieved some understanding of the way in which the proportion of primes decreases. (This understanding draws on a branch of mathematics that appears totally unrelated to the theory of positive integers but is concerned, instead, with the continuous variation of one quantity with respect to another.) There remain significant lacunae, however, and a proof that the Riemann Hypothesis is true would help to fill them. It might also have implications for both physics and communications technology.*
>
> ### *P vs NP*
>
> *A second problem, which may be the most accessible, is called the P v. NP Problem. Computer scientists distinguish two types of task that can be undertaken by a computer. Tasks of type P can be undertaken 'efficiently'. Tasks of type E, by contrast, require a certain amount of ineliminable slog: it is impossible for a computer to carry out even a very simple task of type E without taking many more steps than there are atoms in the known universe.*

[29] Review in the London review by A.W. Moore Vol. 26 No. 14-22 July 2004 The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time by Keith Devlin, Vol. 26 No. 14 - 22 July 2004

But there is a third type of task, type NP, which includes most of the big tasks that industry and commerce would like computers to be able to do. A task of type NP can be undertaken efficiently by a computer as long as, at certain critical stages at which the computer requires the answer to a question, it is given the answer rather than having to work it out for itself. (Of course, this is primarily of theoretical interest. In practice, there would always be the question of where the answer came from.) If a computer were blessed with this facility, would the range of tasks that it could undertake efficiently be significantly increased? One would think so. But perhaps not. Perhaps all tasks of type NP are in fact of type P. That is, perhaps the efficiency that would accrue from these ready-calculated answers can accrue anyway, from suitably clever programming. The P v. NP Problem is to determine whether or not this is so. A proof that it is so would have repercussions for industry, commerce and internet security.

A third problem, the Poincaré conjecture which was posed in 1904 by Henri Poincaré was solved in 2010. The solution of this problem seems to have produced results which will inevitably flow into onto areas.

We will not expand on the other problems because of the difficulty in conveying any idea of their significance to all but a few.

3.2 Workshop on Future Directions in Mathematics [30]

A report published in 2012 from a workshop on Future Directions in Mathematics, sponsored by the Office of the Assistant Secretary of Defense for Research and Engineering (ASD(R and E)) gives us an overview of new directions in mathematics and will be the springboard for presenting what is happening in mathematics.

From the Executive Summary:The goals of the workshop were to provide input to the ASD(R and E) on the current state of mathematics research, the most promising directions for future research in mathematics, the infrastructure needed to support that future research. ...... The invited mathematical scientists came from universities and industrial labs, and included mathematicians, statisticians, computer scientists, electrical engineers and mechanical engineers. This group consisted of established

[30] Report from the Workshop on Future Directions in Mathematics, Institute for Pure and Applied Mathematics University of California, Los Angeles February 8, 2012

leaders in various fields of mathematics, and brilliant young researchers representing recent breakthroughs and future directions....

The participants agreed that the main body of the report should consist of 5 sections:

- Drivers for mathematics; i.e., developments outside of mathematics that influenced mathematics over the last decade
- Recent accomplishments; i.e., the major achievements in mathematics over the past decade
- Future directions; i.e., their best prediction for the most important achievements in mathematics over the next decade
- Infrastructure; i.e., the infrastructure requirements for supporting those future achievements
- International developments; i.e., a critical comparison of mathematics in the US and around the world.

We will expand on the first three dot points in the following paragraphs.

The Drivers for mathematics

The participants identified 6 main drivers for mathematics in the twenty first century:

- Computing
- Big data
- Increasing complexity
- Uncertainty and risk
- Rise of interdisciplinary research
- Connectedness

Recent accomplishments

The participants identified important accomplishments over the last decade in 7 areas of mathematics

- Information science
- Discrete mathematics
- Computation
- Bioinformatics
- Optimization and control
- Nanoscale systems
- Partial differential equations and randomness

Future directions

The participants expect to see great progress over the next decade in 7 areas of mathematics:

- Mathematics of physical systems
- Mathematical Modelling
- Simulation
- Information science
- High dimensionality and large data sets
- Imaging
- Computational and statistical reasoning

In the following the explanations of the above dot points are often directly from the report.

3.2.1 Drivers for Mathematics

Mathematics has changed significantly over the last two decades. Within the discipline, there has been a blurring of the line between pure mathematics and applied mathematics, as well as development of interactions between many different mathematical fields. At the same time, interactions between mathematicians (both pure and applied) and scientists have greatly increased. These trends are expected to continue, with important new developments in many fields.

Computing

Much of the recent impetus for mathematics has come from the growth in computing power and in computer networking. The increase in both the availability and the need for computing power has led to development of new computer architectures, new computational algorithms and new computational models. Sophisticated mathematics has been required for analysis of these new computational models and algorithms. One example is quantum computing – the understanding of which has relied on insights from mathematics to a greater extent than classical computing. Meanwhile, though, the state-of- the-art classical algorithms, even for purely combinatorial problems, have developed an increasingly analytic flavour, so that continuous mathematics has become more and more widespread in the theory of classical computing as well. Many of the other drivers listed above are instigated or enabled by the growth in computing and networking.

Big Data

See Section 4

Increasing complexity

Complexity and its significance have been growing in systems of all types. This includes engineered systems such as the internet and the power grid; social systems such as the financial system and social networks; natural systems such as the global climate system and the global ecological system; and mathematical systems such as large scale graphs and multiscale models. This increased complexity requires new approaches to mathematical modelling, new computational algorithms, and new methods of analysis.

Uncertainty and risk

Uncertainty is an essential feature of many systems. While it has always been present, uncertainty has become more important in many systems because of their increased size and complexity. Examples include: climate for which uncertainties in modelling (e.g., for clouds and for ocean/atmospheric interactions) can make a significant difference in the severity of global warming predictions; the power grid in which small random disruptions can cascade into large systemic power outages; and financial systems, in which risk has long been a central feature but recent examples of financial contagion and collapse demonstrate that current models of risk and techniques for managing risk are not sufficient. (See also section 4)

Rise of interdisciplinary research

Many of the most interesting developments in science and technology over the last two decades have been interdisciplinary, including nanoscience, bioinformatics and sustainability (e.g., climate, environment, energy).

Connectedness

Highly connected systems are proliferating, such as networked sensors and actuators, mobile devices and distributed surveillance. They share a communications resource (and sometimes a computing resource) and occupy a dynamically evolving environment including varying priorities of needs.

3.2.2 Recent accomplishments

The expansion of the dot points on recent accomplishments becomes quite technical in its language and we only outline a few. What is clear is that there is a blending of mathematical ideas and techniques. Statistical and probabilistic techniques blend with mathematical analysis. Graph theory, forming algorithms and seeking more efficient algorithms is a recurrent idea. The use of partial derivatives tat build from elementary calculus is still prominent.

INFORMATION SCIENCE

Compressed Sensing

Many large datasets have a sparse representation, in the sense that the number of significant features in the data is much smaller than the size of the dataset. Compressed sensing provides a method to take advantage of this sparsity; for example, a method for reconstruction of the full large dataset from a number of measurements that is only logarithmically larger than the small number of features. By its combination of harmonic analysis, probability and numerical analysis, compressed sensing epitomizes the new applications of pure mathematics and the interdisciplinary interactions between areas of mathematics. More generally, compressed sensing has inspired the use of sparsity and order reduction in many other areas of mathematics, science, and engineering.

Partial differential equations and stochastic methods for imaging and animation

Automated analysis of images and extraction of features have become important because of the proliferation of imaging. Great progress in image analysis and manipulation of images has been achieved through the use of variational principles and PDEs. For example, PDEs that sharpen interfaces have been used for denoising, and numerical methods based on PDEs have proven to be robust and stable. More recently, non-PDE methods, such as the method of nonlocal means, have been surprisingly successful at image restoration and other imaging tasks such as dictionary based processing. For problems such as oil exploration, the resulting images are dominated by noise; e.g., due to fluctuations in the material properties such as the sound speed. Methods based on stochastic analysis have been successful at extracting features in these problems.

Efficient search algorithms, using graph theory [31]

The emergence of the Web and online social systems give graph theory an important new application domain. We live in a highly interconnected world. Small world phenomena have fascinated the public imagination for many years. With the increased ease of communication, this is true today more than ever. Recent study has found that the average distance between members of the social network Facebook, that contains roughly half of the world's population above the age of 13, is 4.74. This connectedness and network structure is not limited to our social network, but affects almost all aspects of our lives, including financial networks and the web. Large networks increasingly tightly connect our technological and economic systems. Such networks provide great opportunities, but also provide great challenges. The sheer size of these networks makes it hard to study them.

DISCRETE MATHEMATICS

[31] See AMSI publication and video on Google Page Rank http://www.amsi.org.au/ESA_Senior_Years/media/Google_PageRank.htm

Prime progressions[32]

This is particularly interesting as it refers to the work of the Australian Field's Medallist Terry Tao who was a participant in this meeting.

> *God may not play dice with the universe, but something strange is going on with the prime numbers.* (Paul Erdös, 1913-1996)

The field referred to as additive/arithmetic combinatorics came to international attention with the celebrated Green-Tao theorem on long arithmetic progressions of primes in 2004. A prime progression is a sequence of prime numbers $p_1, p_2, \ldots, p_L$ such that the difference between any two successive primes $p_i$and p_{i+1} is equal to the same number K; i.e.,$p_{i+1} - p_i = K$ for any i. The Green-Tao Theorem says that for any L and M (no matter how large), there is a prime progression with parameters K and L, for some $K > M$. The proof of this result used number theory, combinatorics, probability and analysis. Since 2004 the ideas and techniques have spread in many different directions, touching not only on number theory, combinatorics, harmonic analysis and ergodic theory, but also on geometric group theory, theoretical computer science, model theory, point set topology and other fields. A better name for the field might now be 'approximate structures'. In any case, it is clear that the algebraic, combinatorial and probabilistic aspects of very large structures (e.g., graphs, networks and matrices) have become a topic of great interest and wide applicability.

We also note that two other longstanding conjectures for primes have recently been resolved[33]

The first of the two latest results concerns what is known as the twin prime conjecture, which posits that primes just two apart - 3 and 5, 11 and 13, 17 and 19, 41 and 43, and so on - continue to appear indefinitely. Numerical searches appear to confirm the conjecture and mathematicians generally believe that it is true.

But on May 13, Yitang (Tom) Zhang of the University of New Hampshire, Durham, announced a proof that there are infinitely many prime numbers separated by less than 70,000,000.

Clearly, 70,000,000 is not two, but Zhang's is the first result to establish any finite bound at all.

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[32] For a very practical application of primes see the AMSI publication and video http://www.amsi.org.au/ESA_Senior_Years/media/Cryptography.html

[33] Jonathan Borwein and David Bailey, 3 June 2013, The Conversation, You wait ages for number theory results, then two come at once

By the end of May 2013[34], mathematicians had uncovered simple tweaks to Zhang's argument that brought the bound below 60 million. A May 30 blog post by Scott Morrison of the Australian National University in Canberra ignited a firestorm of activity, as mathematicians vied to improve on this number, setting one record after another. By June 4, Terence Tao of the University of California, Los Angeles, a winner of the Fields Medal, mathematics' highest honour, had created a 'Polymath project,' an open, online collaboration to improve the bound that attracted dozens of participants.

For weeks, the project moved forward at a breathless pace. 'At times, the bound was going down every thirty minutes,' Tao recalled. By July 27, the team had succeeded in reducing the proven bound on prime gaps from 70 million to 4,680.

Now, a preprint posted to arXiv.org on November 19 by James Maynard, a postdoctoral researcher working on his own at the University of Montreal, has upped the ante. Just months after Zhang announced his result, Maynard has presented an independent proof that pushes the gap down to 600. A new Polymath project is in the planning stages, to try to combine the collaboration's techniques with Maynard's approach to push this bound even lower.

'The community is very excited by this new progress,' Tao said.

Polymath is available on line at
http://michaelnielsen.org/polymath1/index.php?title=Bounded_gaps_between_primes

The second result concerns what's known as the Goldbach conjecture, which in its strong form is that every even number greater than two is the sum of two primes, and in its weak form that every odd number greater than five is the sum of three primes. Note, for instance, that $13 = 3 + 5 + 5$ and $36 = 17 + 19$.

As with the twin-prime scenario, these conjectures have been studied in great detail, both mathematically and numerically, and are generally thought to be true, but there had been no proof of either.

On the same day as the twin-prime announcement, Harald Helfgott, a 35-year-old mathematician at the École Normale Supérieure in Paris, announced a proof of the weak Goldbach conjecture.

Lattice based cryptography

Over the past decade, a new type of public-key cryptography has emerged, whose security is based on the presumed intractability of finding a 'good' basis for a

[34] Together and alone, closing the prime gap, Quanta Magazine November 2013

high-dimensional lattice (where 'good' means that the basis vectors are short).

Compared to currently-popular publickey cryptosystems, such as RSA and Diffie-Hellman, lattice-based cryptography promises advantages. as far as anyone knows today, lattice-based cryptography would remain secure even against attacks by quantum computers. Second, the encryption and decryption functions in lattice-based cryptography are remarkably simple mathematically – basically amounting to a matrix-vector multiply, plus addition or subtraction of a small error term. The study of lattice-based cryptography has led to many beautiful mathematical ideas – and strangely, understanding the security even of 'classical' lattice-based systems has often required arguments and assumptions involving quantum computation.

Under the heading of **Discrete mathematics** other topics expanded on were **Deterministic primality testing** and **The Langlands program**

PARTIAL DIFFERENTIAL EQUATIONS AND RANDOMNESS

Poincaré conjecture[35]

In 2002-03, Grigoriy Perelman presented a proof of the Poincaré Conjecture that every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. His proof was based on the Ricci-flow method developed by Hamilton. In spite of some initial controversy about the proof, its correctness and full credit to Perelman are now well settled. This was the first solution to one of the seven Millennium Prize Problems from the Clay Mathematics Institute.

Under the heading of **Partial Differential Equations and randomness** other topics expanded on were **Schramm-Loewner evolution** and **Compactness and regularization in PDEs and statistical physics**

COMPUTATION

Computational mathematics involves mathematical research in areas of science where computing plays a central and essential role, emphasising algorithms, numerical methods, computational discrete mathematics, including number theory, algebra and combinatorics, and related fields such as stochastic numerical methods.

It is appropriate here to mention the work of Arthur Engel a German mathematician who has been interested in teaching mathematics in schools. Echoing Klein's title he wrote 'Elementary Mathematics from an Algorithmic Standpoint'[36]. He also wrote a

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[35] Mathematics: The New Golden Age, Keith Devlin,Penguin 1988

[36] Elementary mathematics from an algorithmic standpoint, Arthur Engel, Keele Mathematical Education

much more easily accessible book 'Exploring Mathematics with your computer [37] .Engel had the insight that with computers and calculators widely available, students would no longer accept rote learning of algorithms that could be executed mechanically. He proposed that mathematics in schools should instead focus on the concept of the algorithm, and the syllabus should be completely revised to take an "algorithmic standpoint". His proposed approach would focus on construction and testing of algorithms rather than their execution.

Fast Multipole Methods and analysis-based fast algorithms

The last decade has seen the emergence of analysis-based fast algorithms as a broad generalization of the Fast Multipole Method (FMM) (developed in the 1980s for electrostatic and acoustic applications). FMM-based schemes are now in wide use in stealth modelling, in the chip industry, and in quantum chemistry. Previously intractable problems with millions or billions of unknowns can now be solved routinely using FMM-accelerated iterative schemes.

Shor's algorithm and quantum information science

In 1994, Shor discovered a remarkable algorithm for factoring integers efficiently using a quantum computer. The factoring problem is important not only because it resisted an efficient classical solution for millennia, but because since the 1980s, its conjectured hardness has been the basis for almost all cryptography used on the Internet. Shor's algorithm provided the first convincing evidence that quantum computers could actually help in solving a practical problem-other than the simulation of quantum mechanics itself.

Randomized methods

Compressed sensing and similar computations depend on randomized numerical linear algebra methods. This is not Monte Carlo; randomness is required so that the numerical basis elements have nontrivial intersection with the basis elements in the sparse representation. This has opened up a new field of numerical linear algebra and many open problems remain, such as construction of high order randomized methods.

BIOINFORMATICS

Publications, 1984

[37] Exploring Mathematics with Your Computer. Washington, DC: Mathematical Association of America 1993

Sequencing algorithms for genomics

Dramatic advances in massively parallel sequencing technology during the past few years have resulted in a growth rate of genomic sequence data that is faster than Moore's law [38]. Our ability to extract useful information from this massive accumulation of fundamental genetic data hinges on the availability of efficient algorithms for sequence alignment/assembly, and statistical methods for the modelling of sequence variations and the correlation with phenotypic data.

OPTIMISATION AND CONTROL

Game theoretic management of networks

In settings ranging from the Internet architecture to global financial markets, interactions happen in the context of a complex network. The most striking feature of these networks is their size and global reach: they are built and operated by people and agents of diverse goals and interests, i.e., diverse socioeconomic groups and companies that each try to use a network to their advantage. Much of today's technology depends on our ability to successfully build and maintain systems used by such a diverse set of autonomous users, and to ensure that participants cooperate despite their diverse goals and interests. Such large and decentralised networks provide amazing new opportunities for cooperation, but they also present large challenges. Game theory provides a mathematical framework that helps us understand the expected effects of interactions, and develop good design principles for building and operating such networks. In this framework we think of each participant as a player in a non-cooperative game. In the game each player selects a strategy, selfishly trying to optimise his or her own objective function. The outcome of the game for each participant depends, not only on his own strategy, but also on the strategies chosen by all other players. Mechanism theory deals with the setting of objective or payoff functions for the players of a game. These rules inherently reward efficient behaviour and punish errant actions by individual players. Game theory more widely deals with the concepts of cooperative or competitive dynamics and of equilibria and strategy. This emerging area is combining tools from many mathematical areas, including game theory, optimisation, and theoretical computer science.

Nanoscale systems

Nanoscale systems present a number of important challenges to mathematics and science. They are maximally complex in that they involve both quantum and classical

[38] Moore's Law is a computing term which originated around 1970; the simplified version of this law states that processor speeds, or overall processing power for computers will double every two years.

physics, as well as 11 continuum, atomistic and n-body phenomena The term nanoscale refers to structures that can be measured in nanometers (one billionth of a meter or 0.000000001 meters), from the atomic scale (angstroms) to the cellular scale (tens of microns.) A single nanometer is approximately one hundred thousand times smaller than the thickness of a human hair. Many outstanding problems in nanoscale science concern phenomena in the size-range between 1 and 100 nanometers, and these problems are a major focus of CNS efforts.

Nanoscale Systems are a set of nanoscale components or structures working together to serve a purpose or function. These systems may be in the form of materials, sensors, devices or experimental constructions for the measurement of fundamental physical, chemical or biological properties. Improved understanding of phenomena on the nanoscale is crucial for many areas of science and technology. Progress in nanoscale systems will be essential for advances in medicine, biology, and environmental science, as well as in materials science and computer technology.

3.3 Computing and pure mathematics

In the above we talked about the four-colour problem and its proof and the verification of the proof using computer techniques. Another very old problem has recently been solved and the proof accepted. This is the problem of stacking spheres[39].

The problem is a puzzle familiar to greengrocers everywhere: what is the best way to stack a collection of spherical objects, such as a display of oranges for sale? In 1611 Johannes Kepler suggested that a pyramid arrangement was the most efficient, but couldn't prove it.

Thomas Hales first presented a proof that Kepler's intuition was correct in 1998. Although there are infinite ways to stack infinitely many spheres, most are variations on only a few thousand themes. Hales broke the problem down into the thousands of possible sphere arrangements that mathematically represent the infinite possibilities, and used software to check them all.

But the proof was a 300-page monster that took 12 reviewers four years to check for errors. Even when it was published in the journal Annals of Mathematics in 2005, the reviewers could say only that they were "99 per cent certain" the proof was correct.

In 2003, Hales started the Flyspeck project, an effort to vindicate his proof through

[39] http://www.newscientist.com/article/dn26041-proof-confirmed-of-400yearold-fruitstacking-problem.html.VAESF5W9KHk

formal verification. His team used two formal proof software assistants called Isabelle and HOL Light, both of which are built on a small kernel of logic that has been intensely scrutinised for any errors - this provides a foundation which ensures the computer can check any series of logical statements to confirm they are true.

Recently, the Flyspeck team announced they had finally translated the dense mathematics of Hale's proof into computerised form, and verified that it is indeed correct.

Experimental Mathematics - David Bailey/Jonathan Borwein/ and others

Over the past 25 years, experimental mathematics has developed as an important additional arrow in the mathematical quiver. Many mathematical scientists now use powerful symbolic, numeric and graphic (sometimes abbreviated "SNAG") computing environments in their research, in a remarkable departure from tradition. While these tools collectively are quite effective, challenges remain in numerous areas, including:

- rapid, high-precision computation of special functions and their derivatives;
- user-customisable symbolic computing;
- graphical computing;
- data-intensive computing;
- large-scale computing on parallel and GPU architectures (including algorithm and software design for such systems).

In their paper 'Exploratory Experimentation and Computation'[40] David Bailey and Jonathan Borwein argue that computers can be useful even essential in mathematics research but as yet this has not been transferred into every part of the discipline. By experimental mathematics they intend

1. gaining insight and intuition;
2. visualizing math principles;
3. discovering new relationships;
4. testing and especially falsifying conjectures;
5. exploring a possible result to see if it merits formal proof;
6. suggesting approaches for formal proof;
7. computing replacing lengthy hand derivations;

[40] Exploratory Experimentation and Computation, David H. Bailey and Jonathan M. Borwein, Notices of the AMS, Volume 58, Number 10, November 2011

8 confirming analytically derived results.

Of these items, (1) through (5) play a central role, and (6) also plays a significant role for us but connotes computer-assisted or computer-directed proof.

They speak of using such packages as Maple, Mathematica and MATLAB, general purpose programming languages, internet based applications such as Sloane's encycopeaia of integer sequences and internet databases and facilities.

4. Contemporary applications

4.1 .Areas of Mathematics with many immediate applications

As we have seen there are many contemporary applications of mathematics. in this subsection we look at probability, statistics and operations research. In many ways this section is a reiteration of the ideas of previous sections.

4.1.1 Probability

Probability is an area of growing importance. Its applications are many. Some are listed here.

- The probabilistic understanding of processes in **Biology** such as genetic inheritance, evolution, and epidemics, has been essential for scientific progress for more than 100 years. The recent explosion in the amount of data from genome projects and other sources, such as microarray experiments, has led to the need for new probability models to understand both the structure of the data and the underlying biology.
- Impressive progress on problems in mathematical physics has recently been made based on modern probabilistic methods.
- The application of probability to **finance** has revolutionised an industry. In the past twenty years, the creation of multitrillion dollar derivative security markets has facilitated the worldwide flow of capital and thereby enhanced international commerce and productivity. Without the probabilistic models that provide reliable pricing of derivative securities and guide the management of associated risk, these markets could not exist.
- In computer science, randomised algorithms enable the solution of complex problems that would otherwise be inaccessible. Probability theory provides an essential framework for mathematically interpreting and predicting the behaviour of complex networks. These include both human designs such as the Internet, power

networks, wireless communication, and modern manufacturing systems, as well as natural geophysical systems such as seismic, climatic and hydrologic systems.

- Stochastic reasoning appears central to understanding how spoken language and visual images are interpreted by the mind and is arguably the dominant approach for machine interpretation, for instance for constructing algorithms to recognise speech, identify objects in images and retrieve information from massive data sets.

  In particular Statistics and Probability are and have always been inextricably linked.

4.1.2 Statistics and big data

The applications of statistics in the world today are many. The following are some of the many areas of application. Agriculture, Animal Population, Astronomy, Biology, Census, Chemistry, Computer Science, Demography, Ecology, Economics, Education, Engineering, Epidemiology, Finance, Forestry, Genetics, Government, Health Science, Insurance, Law, Manufacturing, Marketing, Medical Clinical Trials, Medicine, National Defence, Pharmacology, Physics, Political Science, Psychology ,Public Health, Safety, Science Writing and Journalism, Sociology, Sports, Survey Methods, Telecommunications, Transportation, Zoology

Big Data[41]

The following comes from the abstract of the paper by Jianqing Fan, Fang Han and Han Liu. The first and last of these authors are from the Department of Operations Research and Financial Engineering, Princeton University and the second from the Department of Biostatistics, Johns Hopkins University.

> *Big Data bring new opportunities to modern society and challenges to data scientists. On the one hand, Big Data hold great promises for discovering subtle population patterns and heterogeneities that are not possible with small-scale data. On the other hand, the massive sample size and high dimensionality of Big Data introduce unique computational and statistical challenges, including scalability and storage bottleneck, noise accumulation, spurious correlation, (incidental endogeneity) and measurement errors. These challenges are distinguished and require new computational and statistical paradigm....*

Areas of sources of big data include:

[41] Challenges of Big Data analysis, National Science Review Advance Access published February 6, 2014 Jianqing Fan, Fang Han and Han Liu

Genomics Genomics is a discipline in genetics uses probability and statistics (bioinformatics) to sequence, assemble, and analyse the function and structure of genomes(the genetic material of an organism).

Many new technologies have been developed in genomics These technologies allow biologists to generate hundreds of thousands of datasets and have shifted their primary interests from the acquisition of biological sequences to the study of biological function. Thee availability of massive datasets sheds light towards new scientific discoveries. For example, the large amount of genome sequencing data now make it possible to uncover the genetic markers of rare disorders and associations between diseases and rare sequence variants.
Neuroscience Many diseases, including Alzheimer's disease, Schizophrenia, ,Depression and Anxiety, have been shown to be related to brain connectivity networks. Understanding the hierarchical, complex, functional network organization of the brain is a necessary first step to explore how the brain changes with disease. Rapid advances in neuroimaging techniques provide great potential for the study of functional brain networks, i.e. the coherence of the activities among different brain regions. The data from such techniques are massive and very high dimensional.
Economics and finance Corporations are adopting the data-driven approach to conduct more targeted services, reduce risks and improve performance. They are implementing specialized data analytics programs to collect, store, manage and analyse large datasets from a range of sources to identify key business insights that can be exploited to support better decision making. For example, available financial data sources include stock prices, currency and derivative trades, transaction records, high-frequency trades, unstructured news and texts, consumers' confidence and business sentiments buried in social media and internet, among others. It requires professionals who are familiar with sophisticated statistical techniques.
Other applications Social media and the Internet contain massive amount of information on the consumer preferences, leading economics indicators, business cycles, and the economic and social states of a society. It is anticipated that the social network data will continue to explode and be exploited for many new applications. Several other new applications that are becoming possible in the Big Data era include:

- Personalised services. With more personal data collected, commercial enterprises are able to provide personalised services adapt to individual preferences. For example, Target (a retailing company in the United States) is able to predict a customer's need by analysing the collected transaction records.
- Internet security. When a network-based attack takes place, historical data on network traffic may allow us to efficiently identify the source and targets of the attack.

- Personalised medicine. More and more health related metrics such as individual's molecular characteristics, human activities, human habits and environmental factors are now available. Using these pieces of information, it is possible to diagnose an individual's disease and select individualised treatments.
- Digital humanities. Nowadays many archives are being digitised. For example, Google has scanned millions of books and identified about every word in every one of those books. This produces massive amount of data and enables addressing topics in the humanities, such as mapping the transportation system in ancient Roman, visualising the economic connections of ancient China, studying how natural languages evolve over time, or analysing historical events.

4.1.3 Operations Research

Operations Research (O.R.) is a discipline that deals with the application of advanced analytical methods to help make better decisions. The terms management science and analytics are sometimes used as synonyms for operations research.

Employing techniques from other mathematical sciences, such as mathematical modelling, statistical analysis, and mathematical optimisation, operations research arrives at optimal or near-optimal solutions to complex decision-making problems.

Operations research overlaps with other disciplines, notably industrial engineering and operations management. It is often concerned with determining a maximum (such as profit, performance, or yield) or minimum (such as loss, risk, or cost.)

Operations research encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency, such as simulation, mathematical optimisation, queuing theory, Markov decision processes, economic methods, data analysis, statistics, neural networks, expert systems, and decision analysis. Nearly all of these techniques involve the construction of mathematical models that attempt to describe the system.

Because of the computational and statistical nature of most of these fields, O.R. also has strong ties to computer science. Operations researchers faced with a new problem must determine which of these techniques are most appropriate given the nature of the system, the goals for improvement, and constraints on time and computing power.

The major sub-disciplines in modern operations research are:

- Computing and information technologies
- Environment, energy, and natural resources
- Stochastic models
- Financial Engineering

- Manufacturing, service science, and supply chain management
- Marketing Science modelling and public sector work
- Revenue management .

4.2 Some areas where mathematics is being applied today

All the way through this paper we have spoken of the applications of mathematics. The public is not generally aware of where it is being used. We have spoken briefly about the interaction between computing and mathematics. It is also used by firms such as Google.

There is a wide variety of Mathematics used at Google. For example Linear Algebra in the PageRank algorithm, used to rank web pages in search results. Or Game Theory, used in ad auctions, or Graph Theory in Google Maps. At Google there are literally dozens of products which use interesting Mathematics. These are not just research prototypes, but real Google products; in which Mathematics play a crucial role. The interested reader can see more at the GOOGLE reference[42] and at the AMSI site [43]

We look at several further uses of mathematics today. These are biology, risk management, cryptography, crystallography and meteorology. There are many many more.

4.2.1 Biology

There has been an explosion of knowledge in the life sciences over the past twenty years. At the centre of this explosion is the use of mathematics and statistics. These advances have expanded use of mathematics and statistics beyond the traditional fields of physical science and engineering. Doctors and scientists hope to use our genetic information to diagnose, treat, prevent and cure many illnesses. This knowledge will eventually lead to more effective medicines and treatments. Biology is in dramatic flux due to a surge of new sources of data, access to high-performance computing, increasing reliance on quantitative research methods, and an internally driven need to produce more quantitative and predictive models of biological processes. The growing infusion of mathematical tools and reasoning into biology may therefore be expected to further transform the life sciences during the decades ahead. This transformation will have profound effects on all areas of basic and applied biology. There is a discussion

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[42] http://www.javiertordable.com/files/MathematicsAtGoogle.pdf

[43] http://www.amsi.org.au/ESA_Senior_Years/media/Google_PageRank.html

about genetics and gene mapping in particular at the AMSI site. [44]

4.2.2 Risk management[45]

Risk management can be seen as a core competence of an insurance company or a bank its While risk management has thus always been an integral part of the banking and insurance business, recent years have witnessed a large increase in the use of quantitative and mathematical techniques. Even more, regulators and supervisory authorities nowadays even require banks to use quantitative models as part of their risk management process. Given the random nature of future events on financial markets, the field of stochastics (probability theory, statistics and the theory of stochastic processes) obviously plays an important role in quantitative risk management. In addition, techniques from convex analysis and optimization and numerical methods are frequently being used. In fact, part of the challenge in quantitative risk management stems from the fact that techniques from several existing quantitative disciplines are drawn together. This also requires the ability to interact with fellow workers with diverse training and background.

4.2.3 Cryptography [46]

Encryption plays a crucial role in the day-to-day functioning of our society. For example, millions of people make purchases on the internet every day. Each time you submit your credit-card details online, there is a risk that this information may be stolen. So how can the information be sent securely? A shopper's credit-card details need to be encrypted before they are transmitted over the internet, and so the method of encryption needs to be made public. But the method of decryption should be known only to the bank that is processing the payment. For all of the ciphers in use before RSA, the methods of encryption and decryption were known to both the sender and the receiver of the message. With RSA, the instructions for how to encrypt a message can be made public, without compromising the security of the method of decryption. This was the big breakthrough that came with RSA encryption.

Today new encryption methods are being developed - besides number theory these are requiring the use of structures such as rings and lattices.

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[44] http://www.amsi.org.au/ESA_Senior_Years/media/Gene_Mapping.html

[45] Mathematics in Financial Risk Management Ernst Eberlein, Rüdiger Frey, Michael Kalkbrener, Ludger Overbeck, http://www.stochastik.uni-freiburg.de/ eberlein/papers/risk-management-survey-final2.pdf

[46] AMSI resources, RSA encryption http://www.amsi.org.au/ESA_Senior_Years/media/Cryptography.html

4.2.4 Crystallography[47]

Crystallography is the science that examines the arrangement of atoms in solids. It has always had substantial input from mathematics.[48]

Mathematical chrystallography provides one of the most important applications of elementary geometry to physics. The relationship between geometry, symmetries and group theory date back to the nineteenth century.

Far from having exhausted its research potentiality, Mathematical and Theoretical Crystallography is today facing new challenges, not only in the very classical field of group theory (magnetic groups, chromatic groups, N-dimensional groups) and its applications (phase transitions, polymorphism and polytypism, twinning, bicrystallography, ferroic crystals), but also in several directions that previously were less strongly perceived as being directly related to crystallographic and crystal-chemistry problems, such as graph theory, combinatorial topology, number theory, discrete geometry, diffraction theory, etc. The development of mathematical and theoretical crystallography will strengthen the interaction between crystallographers, mathematicians and materials scientists and will definitely contribute to the recognition of crystallography as an interdisciplinary science.

4.2.5 'Traditional' mathematical modelling - meteorology

There are many areas of our lives where mathematical modelling is used. One of the most important areas is modelling the behaviour of weather and the oceans. Especially when we are all interested in the question of climate change mathematics is at the forefront of answering questions in these areas. The mathematics has changed a great deal but it is interesting that a lot of the ideas used today date back to a book by Lewis Fry Richardson[49]. The first edition of this book, published in 1922, set out a detailed algorithm for systematic numerical weather prediction. The method of computing atmospheric changes, which he mapped out in great detail in this book, is essentially the method used today. He was greatly ahead of his time because, before his ideas could bear fruit, advances in four critical areas were needed: better understanding of the dynamics of the atmosphere; stable computational algorithms to integrate the equations; regular observations of the free atmosphere; and powerful automatic

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[47] International Union of Crystallography Commission on Mathematical and Theoretical Crystallography http://www.crystallography.fr/mathcryst/

[48] Introduction to Geometry, H S M Coxeter, John Wiley 1969

[49] Weather Prediction by Numerical Process,2nd Edition, Part of Cambridge Mathematical Library ,Lewis Fry Richardson, August 2007

computer equipment. Over the ensuing years, progress in numerical weather prediction has been dramatic. Weather prediction and climate modelling have now reached a high level of sophistication, and are witness to the influence of Richardson's ideas.

5. Possible implications for mathematics education

How do the areas of discussion in the above sections influence what is happening in the classroom? There is little doubt that the above says that the nature of mathematics has changed over the past two centuries and we can expect it to continue to change. The mathematics in the school classroom has not changed at the same rate. There are evolutionary changes taking place in the Victorian curriculum but it is slow.

Technology is changing our society in every way and it would naive to think that it is not effecting education but the change has been slow. The first changes date back to the introduction of calculators in the 1970's and the introductory use of computers. Computing for schools was introduced and the book 'Computing for schools using MINITRAN' by K. McR. Evans and R.D. Money [50] was written and the first steps were taken. This book was listed as preliminary reading for students undertaking the General Mathematics computer option. Many schools introduced programming into their Mathematics courses. New software appeared, spreadsheets appeared in schools and by the mid 1990's were being used extensively. We discuss technology further in the following section.

The content of Victoria's 'higher level' courses, 'Mathematical Methods' and 'Specialist Mathematics' have many similarities to the courses of 1972 and with a large overlap back to 1944 and before. This is not necessarily a bad thing but we must be asking the questions about whether our pathways are appropriate in the twenty first century. We will explore that development of our senior courses in the following section.

5.1 Some views on what we should be aiming for

5.1.1 Australian Assocation of Mathematics Teachers [51]: School mathematics for the 21st century: Some key influences

Mathematics for the 'knowledge economy' There is a strong argument that mathematics is increasingly important in our society. It is clear that the pervasive

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[50] Computing for schools using MINITRAN / K. McR. Evans, R.D. Money, Adelaide : Rigby, 1971.

[51] http://www.aamt.edu.au

technologies of our times are, and will continue to be, substantially based on, and enablers of, mathematics. Those developing technologies need, of course, to be highly mathematically competent. On the broader level our society is very much driven by data and analyses of mathematical models that result from the use of the technologies. Everyone needs mathematical skills and capabilities.

On the other hand, however, these technologies effectively ’submerge’ the popular perception of what constitutes ’mathematics’ - mathematics seems nowhere near as important as it used to be. No-one needs to be able to manually do a whole range of things such as simple and now, very complex, calculation. These can be automated. This paradox is resolved if we consider what we mean by mathematics. ’Low level’ skills that can be more accurately and efficiently done by a machine are certainly much less relevant in use and can no longer be supported as the key outcomes of school mathematics for their own sake. The 21st century requires mathematics of a higher order for citizens to be able to understand, work with and create mathematical models that are accessible and powerful in the context of current and emerging technologies. As a result, the important mathematics in schooling should be about this sort of mathematics.

That is not to say that formerly important ’lower level’ skills are not important. In some, perhaps many cases they are, insofar as they are integral to being able to work with powerful mathematical tools, technologies and techniques. For example, the emergence of mental computation as an important component in young children’s facility with number has clear practical uses. But mental computation is also important through its capacity to deepen understanding of how numbers ’work’ and therefore critical to fluent use of mathematical models of all sorts.

There needs to be a major rethinking of school mathematics in the light of all this. Its purposes have to be in line with the mathematical needs of young people and their futures. School mathematics needs to maximise its fit with the expectations of the wide range of other stakeholders such as government, higher education, business and industry.

Hence mathematics content needs to be selected for clear reasons that link to the nature of the mathematics that citizens need in and from schooling, and into the future. These reasons do not include ’we have always done it.’ Apparently heretical questions like ’Should the kind of ’procedural’ calculus that has been the pinnacle of achievement in school mathematics in the 20th century remain so in the 21st?’ and ’Does the emphasis on algebraic skills serve students’ and the society’s needs?’ need to be debated.

5.1.2 Keith Devlin at the Stockholm conference

Keith Devlin in his Stockholm talk listed the following for innovative mathematical thinkers

- Think outside the box
- Find/adapt existing methods/techniques for novel situations
- Find new approaches/methods/techniques for new problems
- Collaborate with others - work in multidisciplinary teams
- Communicate well
- Broad sense of the scope, power, and limitations of mathematics
- Good (not necessarily stellar) mathematical ability
- Ability to quickly master new mathematical techniques

Must have:

- A broad sense of the scope, power, and limitations of mathematics
- Good (not necessarily stellar) mathematical ability
- Ability to quickly master new mathematical techniques

Finally he put forward the following

- For over two thousand years, books were the only means to store and disseminate information to society = a technology limitation! Textbook delivery has shaped our view of what mathematics is and how to do it.
- Mathematics is about doing, not knowing.
- Mathematical thinking is primarily a way of thinking about entities, issues, and problems in the world.
- Though much advanced mathematics is linguistically defined, a lot of mathematical thinking can be done (or learned) without formal notation (symbols), or perhaps with new representations

5.1.3 Functional versus specialist

In the recent publication, 'Teaching Mathematics: Using research-informed strategies'[52] Professor Mike Askew summarises some pragmatic thoughts of Peter Sullivan on the role of mathematics.

[52] Sullivan, Peter, "Teaching Mathematics: Using research-informed strategies" (2011). http://research.acer.edu.au/aer/13

> *Sullivan frames his review by tackling head on the issues around the debate about who mathematics education should be for and consequently what should form the core of a curriculum. He argues that there are basically two views on mathematics curriculum - the 'functional' or practical approach that equips learners for what we might expect to be their needs as future citizens, and the 'specialist' view of the mathematics needed for those who may go on a study it later. As Sullivan eloquently argues, we need to move beyond debates of 'either/ or' with respect to these two perspectives, towards 'and', recognising the complementarity of both perspectives.*
>
> *While coming down on the side of more attention being paid to the 'practical' aspects of mathematics in the compulsory years of schooling, Sullivan argues that this should not be at the cost of also introducing students to aspects of formal mathematical rigour. Getting this balance right would seem to be an ongoing challenge to teachers everywhere, especially in the light of rapid technological changes that show no signs of abating. With the increased use of spreadsheets and other technologies that expose more people to mathematical models, the distinction between the functional and the specialist becomes increasingly fuzzy, with specialist knowledge crossing over into the practical domain. Rather than trying to delineate the functional from the specialist, a chief aim of mathematics education in this age of uncertainty must be to go beyond motivating students to learn the mathematics that we think they are going to need (which is impossible to predict), to convincing them that they can learn mathematics, in the hope that they will continue to learn, to adapt to the mathematical challenges with which their future lives will present them*

The information given in the preceding sections of this paper supports such thinking. The need for some mathematical rigour should not be confined to a small section of our school population.

5.1.4 Conrad Wolfam DEECD meeting May 28 2014

Few educationalists deny the importance of the subject, but an increasing number believe that lessons are becoming irrelevant

This should be the golden age of maths in schools. After generations of benign neglect, the subject has been thrust into the media spotlight. Politicians, educators and private companies maintain that effective maths learning is the key to everything from smarter workers and savvier consumers to improved national competitiveness.

And in truth, research does suggest that maths is key. It provides the underpinnings for a problem-solving mindset that is applicable in every sector of society. From research to manufacturing and the service economy, modern enterprises run on computers, whose coding principles are rooted in the rigorous logic of mathematical thinking. Studies presented by children's charity UNICEF show that maths achievers excel - and earn higher salaries globally- in any profession they enter.

Yet many students still flee from maths in the classroom. Engagement is plummeting. Even those who recognise the compelling arguments for advancing their careers regard maths classes as an ordeal.

Conrad Wolfram founded ComputerBasedMath.org (CBM) in 2010 to promote the use of computers in teaching the subject. He believes that people know there is something called mathematics that is important and yet we have this subject in schools that everybody thought was the subject but it isn't. It's disconnected.

Three years ago Conrad gave a TED talk that has since been viewed nearly 900,000 times. The essence of the talk is that although maths requires a combination of problem formulation, abstraction, calculating and communicating, schools remain steadfastly focused on calculating. The alternative is to teach mathematical thinking and hand the drudge work to computers.

He sees parallels between traditional approaches to teaching maths and subjects like Classics. When people stopped needing Greek and Latin in everyday life, schools continued to teach Classics, but as a proxy for learning English grammar. With maths there is a real-world subject but what we are teaching is a proxy and the proxy isn't working that well. He believes that if stick with your current subject it will turn into Classics. Because in the end you can't justify spending billions of dollars a year around the world teaching a subject that basically nobody is using. It will not survive as a general-purpose subject.

So what needs to be done? He believes there are two choices:

- Essentially a new subject starts to replace maths, or
- we change maths into the subject it ought to be.

What might a new maths curriculum include? CBM has a list that includes: mathematical thinking; everyday maths; finding patterns in information; knowing where you are in space; mathematics in the natural world; mathematics in technology; winning; and money maths.

Another important issue is whether those with traditional training will be able to teach the new approach. Finally, and most seriously, the educational assessment community has not signed on yet. Assessment providers are organised to measure calculating skills and maths content as a body of knowledge, not the open-ended understanding that he espouses.

5.2 The role of technology

Professor John Crossley[53] in an earlier paper for the VCAA writes the following

At this point in our history we can use mechanical, or more frequently electronic, devices to perform repetitious tasks or to shortcut them. This has been the case since slide rules were introduced - some might even say, since the abacus was introduced. What is different now is that a multitude of tasks, identical or different, can be performed at, so to say, the touch of a button and very quickly. This change in quantity brings with it a change in quality.

It is easy to deal well with single computations such as finding an n-th root. This is because we can give a clear and explicit process for finding such a root and a proof that it is correct. On the other hand we do not seem to be able to deal as well or as adequately with long sequences of computations. The very complexity of some computations is too much for us to grasp. We do not seem to be able to handle such sequences in a way that is satisfactory enough from a formal point of view.

In order to equip the next generation adequately we need to give them sufficient understanding of the individual processes and also sufficient understanding of sequences of processes so that they can make informed judgments about the reliability of the software and hardware that they will employ. This cannot start too early.

As well as examples of their usefulness, examples can easily be given where there are problems: try taking the square root of a number over and over again on (different) handheld calculators.

Let us also be thoughtful about when it is appropriate to use computers or calculators. We use them when we do repetitious work. When we need to do a calculation a thousand times, or even weekly or daily, it is foolish to use pen and paper: a computer or calculator is more reliable and less stressful. However, for a one-off, intricate calculation it will usually be easier to just do it, rather than to write, test and run a computer program for it. This is one issue involving scale, here meaning the number of times we repeat a calculation.

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[53] The Essentials of, and for, Mathematics, John N. Crossley, Gazette of the Australian Mathematical Society, Volume 33 Number 3, July 2006.

5.2.1 Algorithms and computer programming

We have spoken about the importance of algorithms in the mathematics of today. It seems more likely that this is again receiving attention in our system. Earlier we also spoke of the work of Arthur Engel in this area of school education and we will trace the development (and demise and rise again) in our system in the final section of this paper.

Algorithms make us think of computations but, as we know to our cost as computer users, algorithms, such as the ones in the software that we use everyday, do not always produce the answer or behaviour that they should. Along with means of computation one also needs means of proof. Together with each algorithm one should have a proof that the algorithm does indeed do what it is supposed to do.

The informatics competition[54] run by the Australian Mathematics Trust gives us an idea of some types of problems available for younger students.

> *Beginning with some number n, you write a line of '*'s by repeatedly applying the following rules:*
>
> - *If n is 0, stop.*
> - *If n is odd, write a single '*' and reduce n by 1.*
> - *If n is even, divide n by 2.*
>
> *For example, if you begin with $n = 3$ then you would proceed as follows. Since 3 is odd, you write a single '*' and subtract one to give $n = 2$. Since 2 is even, you divide by two giving $n = 1$. Finally, since 1 is odd you write another '*' and subtract one. Now $n = 0$ and you stop, having written two '*'s in total. If you begin with the number $n = 77$, how many '*'s do you write in total?*

In the proposed Victorian senior curriculum for mathematics the introduction of the bisection method and Newton's method for solving equations reintroduces this way of thinking into Mathematical Methods Units 1 and 2. In Specialist Mathematics we have Euler's method in Further mathematics, algorithms in graph theory. It is a start and we should be thinking more about this.

One, but only one, of the ultimate goals in the study of structures would be the understanding of the structure of computer programs - in particular the programs that will be composed by the students in the course of their calculations.

Conrad Wolfram points out that calculation dominates the school curriculum and that this should be changing with the increase use of computers. He doesn't say that hand

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[54] http://www.amt.edu.au

calculation should be abandoned but its prominence diminished.

There are now computer languages which are suitable for primary school and are 'modern' in their approach -Scratch[55] is one of these. It is free. Python [56] is also being used in schools.

It is worth noting the following recommendation by National Mathematics Advisory Panel[57].

Recommendation: The Panel recommends that computer programming be considered as an effective tool, especially for elementary school students, for developing specific mathematics concepts and applications, and mathematical problem solving abilities. Effects are larger if the computer programming language is designed for learning (e.g., Logo) and if students programming is carefully guided by teachers so as to explicitly teach students to achieve specific mathematical goals.

6. The Victorian Senior Mathematics Curriculum

There has been substantial work by Maths Educators in this field in recent years in Victoria and in particular the work of Professor Kaye Stacey and her associated researchers at the University of Melbourne. Her work in problem solving and the use of technology have been very influential in the formation of courses. Professor Stacey has been very involved in course design in Victoria during recent decades and she has exerted a very positive influence on what has happened in Victoria. Some of her influential publications are listed as footnotes.[58]

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[55] http://scratch.mit.edu

[56] http://teachers.theguardian.com/teacher-resources/11331/Teaching-Programming-With-Python-Computing-At-School

[57] National Mathematics Advisory Panel. Foundations for Success: The Final Report of the National Mathematics Advisory Panel, U.S. Department of Education: Washington, DC, 2008.

[58]

Kendal, M. and Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. International Journal of Computers for Mathematical Learning 6(2), 143-165.

Pierce, R., Stacey, K. (2004). A framework for monitoring progress and planning teaching toward the effective use of computer algebra systems. International Journal of Computers for Mathematical Learning, 9(1), 59-93.

Kaye Stacey, The place of problem solving in contemporary mathematics curriculum documents, The Journal of Mathematical Behaviour, 24(3),2005

In the following we look at the development of our senior secondary courses.

[59] Until 1960 there was only one university in Victoria, the University of Melbourne.The earliest Victorian Matriculation (University Entrance) Examinations University exams were in 1857. The University of Melbourne set the papers. To begin with there were three mathematics subjects: Arithmetic, Algebra, and Euclid (or Geometry). Books used in the nineteenth century were all published in England and included books such as Todhunter's Trigonometry[60].

The ideas of the Mathematical Association, formed in the Britain in 1897, were influential in the teaching of Mathematics in Victoria and in 1905 a statement was made that all of the mathematics examinations should be set in general accordance with the recommendations contained in the publication: Teaching of Elementary Mathematics: Report of the Committee appointed by the Mathematical Association, (1905)[61]

A highly influential figure in Victorian Mathematics Education was Sir Thomas Cherry. He dominated the mathematical scene in Victoria for thirty-five years. He was educated at Scotch College in Melbourne, the University of Melbourne and completed his PhD at Cambridge University. In 1929 he accepted the Chair of 'Mathematics, Pure and Mixed' at the University of Melbourne. In 1952, separate chairs in Pure and Applied Mathematics replaced this Chair and Cherry was appointed to the Chair of Applied Mathematics, which he occupied till his retirement in 1963. In mathematics he was involved in the Year 12 courses and the examinations throughout his tenure at the university. He established relations with secondary school teachers and delegated work to them. A biographer (Bullen (1967)) [62] writes:

> *By the time I had arrived in Victoria, he had acquired an almost god-like stature among the mathematical teachers of the State, and the quality of preparation of entrants to Melbourne University was streets ahead of that in*

What is mathematical thinking and why is it important, K Stacey - Progress report of the APEC project: Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (II)-Lesson Study focusing on Mathematical Thinking, 2006

[59] Blake, L.J. (Ed). (1973). Vision and Realisation: A Centenary History of State Education in Victoria. Melbourne: Education Department of Victoria.

[60] I Toddhunter, Plane Trigonometry, Macmillan and Co, Cambridge, (1860)

[61] The teaching of elementary mathematics : reports of the committee, Mathematical Association (Great Britain),London : Bell, 1905

[62] Bullen K E, (1967). Biography of Sir Thomas Cherry. Records of the Australian Academy of Science, 1(2), Canberra, Australia.

> *any other Australian State. In spite of current fanfares in some other States, it is doubtful whether school education in Mathematics and Physics will for many years approach the quality it reached in Victoria in Cherry's time.*

Curriculum and examination organisations

From 1857 to 1964 the University of Melbourne set the Matriculation Examinations for mathematics. In 1912 a Schools Board consisting of representatives of the Education Department, independent schools and the university replaced the university's Board of Public Examinations. The Board relinquished control of the matriculation from 1945, and all other examinations from 1965. In July 1964 the Victorian Universities and Examinations Board (VUSEB) took over prescribing the courses and setting the examinations. The newly created Monash University joined with Melbourne in this structure. This all changed in 1979 when a new state structure for curriculum and assessment was established which the universities no longer controlled. This was the *Victorian Institute of Secondary Education.* The Institute was replaced by the *Victorian Curriculum and Assessment Board* (VCAB) in 1986 and this in turn by the *Board of Studies* in 1993 , which was subsequently replaced by the current *Victorian Curriculum and Assessment Authority* (VCAA) and the *Victorian Registration and Qualifications Authority* (VRQA) in 2001.

The Board was responsible for curriculum development and evaluation of courses from Prep to Year 12. It also accredited courses and was responsible for assessment policy from Prep to Year 12. It also administered programs leading to the assessment and certification of the Victorian Certificate of Education for Years 11-12, as does its successor the VCAA.

The Board of Studies was abolished in 2002 and replaced by the Curriculum Assessment Authority which was assigned responsibility for all functions formerly undertaken by the Board of Studies.

Mathematics subjects

Until 1972

In the post second world war years up to 1972 the two high level level courses were *Pure Mathematics* and *Calculus and Applied Mathematics.* A third course *General Mathematics* ran parallel to them. (These subjects are listed as three of the 25 subjects listed by the Professorial Board has prescribed the Matriculation Examination, December, 1944. General mathematics could not be counted with either Pure Mathematics or Calculus and Applied Mathematics as a subject of the Matriculation Examination)

Pure mathematics contained questions on algebra, trigonometry and combinatorics and probability as a side product of combinatorics. The questions still seem familiar such as

- Solve the equation

$$\frac{3^{2x-1}+1}{84} = 3^{x-3}$$

Question 1a, Pure Mathematics Paper 2, 1956

- In a certain country the number of births in each year is $2\frac{1}{2}$ per cent. more than in the previous year, and there were 562,500 births in 1954. What was the number of births in 1937. Assuming that all are still living, find the number of persons aged 17 or less at the end of 1954. (Work to the accuracy permitted by the tables, and do not claim any greater accuracy.

Question 8, Pure Mathematics Paper 2, 1956

This second question is interesting in the light of the discussion on the use of technology given below. Their technology here was 4 figure logarithm tables.

It is evident that any of the questions (with the exception of questions on conics) that were asked on these papers could still be asked within the subjects of the proposed new study designs for mathematics beginning in 2016.

Calculus and Applied mathematics contained calculus questions which are very much like the questions we see today but the applied mathematics contained a lot more than we have today with such topics statics of a rigid body, power, circular motion, simple harmonic motion and energy. There was no probability or statistics

Questions from the 1956 Calculus and Applied paper included.

From first principles find the gradient of the graph of $y = x^2$ at $x = 3$:state clearly the graphical interpretation of the various steps.

Question 1a, Calculus and Applied Paper 1, 1956

At any time t the displacement of a particle moving in a straight line is x. Express its acceleration as a derivative with respect to x.

A particle leaves O with velocity 2 ft/sec, and its acceleration thereafter is equal to $-\frac{1}{6}\sqrt[3]{x}$ ft/sec^2 when its displacement from O is x ft. Find its displacement when it first comes to rest.

Question 7, Calculus and Applied Paper 1, 1956

1972 - 1980

From 1967 until 1971 a new course structure ran parallel to the existing courses of *Pure Mathematics* and *Calculus and applied mathematics* and *General Mathematics*. The new subjects were *Pure Mathematics* (New), *Applied Mathematics* and General Mathematics(New). The new subjects ran unaltered from 1972 to 1980. Despite the small change in names, this was one of the more radical changes in Victorian school mathematics and curriculum changes were implemented which are still with us today. Matrices and complex numbers were introduced, the formal language of sets and functions became a part of every topic . Probability and probability distributions were added to Applied Mathematics and General Mathematics and from 1975 Linear programming questions were regularly included on the General Mathematics paper. Many of these changes were influenced by the New Maths movement which we discuss below.

This was a large step in curriculum innovation and to see its power we only have to observe that much of it is still with us today. It was a timely change.

Richard Tees [63] has commented on other effects of this curriculum change. The quote is from a book review by Colleen Vale[64].

> *Teese discusses the "new mathematics" reform during the 1960s that involved the phasing in of three new subjects called General Mathematics, Pure Mathematics and Applied Mathematics in the Higher School Certificate (HSC) from 1967-71. Teese explains that modern mathematics was concerned with improving the quality of teaching.*
>
> *An increased emphasis on understanding was sought through the use of set theory and the integration of different aspects of mathematics. At this time there was also a growing demand for academic schooling among the middle and upper working classes. General Mathematics attracted increased participation from girls. High socio-economic status (SES) boys who were seeking careers in the professions, such as medicine, participated in General Mathematics at similar levels to the previous decade. Boys from middle and upper working class backgrounds were attracted to Pure Mathematics coupled with Applied Mathematics, the two most demanding mathematics*

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[63] Richard Teese, Academic Success and Social Power: Examinations and Inequality (2nd ed.) (Melbourne, Australia: Australian Scholarly Publishing, 2013,

[64] Colleen Vale Book review of Academic Success and Social Power: Examinations and Inequality(1st edition), Mathematics Education Research Journal, 2001

subjects designed as prerequisites for engineering and science careers. They participated in these two subjects in higher proportions than boys from high SES backgrounds. The performance data five years after the implementation revealed the previously established pattern where students from a high SES background and private schools were less likely to fail and more likely to be awarded high marks than students from low and middle SES schools. This result was the same for all three subjects, but especially for Pure and Applied Mathematics where participation rates were lower for high SES students. Teese argues that the reform process lacked a student perspective and did not address the lack of both resources and mathematics teachers with university mathematics qualifications in government schools. Further, he argues that the reformers did not question the expectations and values set by Matriculation syllabus writers and examiners. The subjects therefore retained an intellectual orientation. The new mathematics, rather than leading to an improvement in understanding, was interpreted as new content, and was taught in the traditional way with poor resources. Examiners continued to require advanced skills in reasoning and explanation. He cites evidence of private school teachers as textbook writers to argue that teachers in the academically established private schools were in the best position to adapt to the new syllabi. Hence the students from families without the cultural and educational capital remained at a disadvantage.

1981 - 1985

In 1981 a new structure was introduced with new subjects that were classified as Group 2 within the Victorian HSC. This coincided with the new organisation, VISE being in control of examinations and curriculum.These subjects were of a more practical nature and had names such as *Mathematics at Work* and *Business Mathematics*. They did not have external examinations and assessment was carried out within the school.

Through the 1970's and early 1980's a new structure was available for schools to offer students at year 12. This was the *Schools Tertiary Entrance Certificate*. Schools offered mathematics inside this structure and it was recognised by VISE. There were considerable conflicts between schools and parents when schools chose to offer only the STC course. For example, Moreland High School in Coburg did this in 1976.

The existing subjects of Pure, Applied and General mathematics were given the classification of Group 1. The examination system continued for these. HSC Group 1 mathematics subjects were also amended to include options. In Applied Mathematics probability disappeared from the core syllabus and was offered as an option in both Pure Mathematics and Applied Mathematics. The other options in Applied

Mathematics were Numerical Mathematics and Computer Applications in Mathematics. Matrices and Complex numbers became an option in Pure Mathematics. In General Mathematics there was also a core option structure. The options were assessed at the school level

1986 - 1990

In 1986 the Group 1 and Group 2 structure continued but now with only two Group 1 subjects, Mathematics B building on the content of Mathematics A. The content of the core did not vary greatly from what we had seen since the substantial change in 1971. There was a large selection of options offered at this stage. Some were much more popular than others. The reason for the popularity was often that the area had been in the subjects of the 1970's. The computer based mathematics option continued. One of interest for the proposed Victorian study designs is the following. It is taken from the VISE, HSC course description for the years 1986 -1990 [65].

> Topic 2: Statistical sampling
>
> *... Statistics play an important role in today's society-in our personal lives, in commerce and in scientific study. Clearly the ability to obtain and correctly interpret statistical data, and to critically appraise the ways in which statistical information is used to support arguments in useful in the modern world.*

1991-1994

This was a period of massive change in the structure of school assessment. It lasted for four years and underwent considerable modification on the way through. We begin with a quote from Colleen Vale.

> *... the introduction of problem solving Quoting from the Blackburn Report he (Teese) identifies the underlying principles of the recommended reforms: 'Any discussion of curriculum must begin by asserting the primacy of essentially common and cultural purposes' In the previous decades the needs of mass secondary education had led to the proliferation of subjects and resulted in a de facto streaming of subjects and hence students. The recommendations of the Blackburn Committee implied that 'the problem of diversity would have to be managed within one certificate. This would not be*

[65] Higher School Certificate Course Description, MathematicsGroup 1, Victorian Institute of Secondary Education, 1985

through formal streams, but through flexible curriculum design and multiple approaches to testing student learning' .

The mathematics course design in the VCE provided a flexible structure and content intended to enable students to engage with a variety of mathematical interpretations and applications. The mathematics subjects were based on major areas of the discipline: Space and Number, Change and Approximation, and Reasoning and Data. A second year of study in each of these was called Extensions, hence there were six subjects that could be taken over two years of the certificate. The mathematics subjects in the new VCE included a new emphasis on the mathematical processes involved in problem solving and investigations. These were assessed through school-based tasks.

Pressure to change the nature and structure of the VCE mathematics during the 1990s, Teese suggests, did not arise from the extremely high failure rates among working class students. Rather, he argues, the changes were instituted because the two major universities expected greater routine facility among their undergraduate students than was evident among the highest achieving VCE students that they selected. Whilst Teese provides evidence to support this argument he could also have referred to the pressure arising from the media's publication of concerns about authentication and the cultural capital of students in private schools.

Mid-way through the 1990s, the revised VCE restored the hierarchical nature of mathematics subjects evident in the previous decades. The mathematics subjects in the revised VCE were Further Mathematics, Mathematical Methods and Specialist Mathematics.

One interesting aspect was the subject *Reasoning and Data* where some very different topics were introduced into the curriculum. Some of these topics had appeared as options in previous courses but here we had a whole course devoted to topics which had been said to be important but could not find a permanent place in the school curriculum. The topics covered included:

- Data description and presentation (The predecessor of the statistics core of Further Mathematics)
- Probability: models for data (This included a study of probability distributions, some of which is now in Mathematical Methods)
- Statistical inference: drawing conclusions from data. (This has again disappeared. It included estimation, confidence intervals, significance testing and χ^2.
- Logic and algebra which included graph theory (some of which has found a place in Further Maths), Boolean algebras. propositional logic, and logic and proof. (It should

be noted that other aspects of this collection have appeared in the year 11 subject General Mathematics but they are not widely incorporated into school courses.

1995 to the present

We will not dwell on the curriculum changes as it has been twenty years of stability after the challenging years from 1991 - 1994. Some of the most substantial changes have come in the use of technology and this is discussed below.

The New Maths in Victoria and the School Mathematics Research Foundation

From 1970 the number of students sitting the university entrance exams for mathematics had increased sufficiently for textbooks to be written within the state. For example There were three books which appeared within a couple of years. This was at the same time that the 'New Maths' was being introduced in both the USA and Australia and there was considerable tension between members of the mathematics community of Victoria and this was reflected to some degree in these new texts. One of the new texts was: School Mathematics Research Foundation, Pure Mathematics (1970)[66]. This book was influential but School Mathematics Research Foundation In 1965 the Mathematical Association of Victoria stressed the need for research into methods of improving the teaching of mathematics in secondary schools. Shortly after this with a grant from the Mathematical Association of Victoria, The School Mathematics Research Foundation was formed with the aim of carrying out and encouraging such research. The first president was Professor Gordon Preston of the new Monash University. This Foundation and the push for the 'New Maths' was centred around Monash University.

The Foundation produced a few publications including a book called Pure Mathematics published in 1970. The authors were a mixture of mathematicians from universities and schoolteachers. In general schools did not adopt this book for any length of time but it has influenced school mathematics in Victoria for the past forty-five years. Generally it was well written and mathematically rigorous. New definitions were not introduced without a rationale for the choice of definition and results were proved wherever possible at this level. The language and concepts were very sophisticated.

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[66] Cameron, N., Clements, M., Green, L. & Smith, G., (1970) School Mathematics Research Foundation: Pure Mathematics , Melbourne: Cheshire.

Technology

The progression of technology and calculating aids has been remarkable. From the 1920's and up to 1978 the main method of calculation in senior school mathematics was the use of logarithm tables. Chris Barling[67] ,in his interesting article on the history of logarithms in this state writes.

> *It may well be that the authorities in Melbourne were in the vanguard of educational practice in this respect, because the commercially published books of four-figure tables that were a familiar feature of school mathematics until the 1970s were all first published in the 1920s, or later. One suspects that the tables provided by the examiners of 1910 were home-grown affairs, possibly compiled at the University expressly to accompany the Public Examinations: if so, it was an innovation of great foresight and its originators should be honoured as pioneers in the ever-accelerating advance of calculating technologies.*

In the examinations of 1953 to 1968 the instruction at the beginning of each paper was that mathematical tables are provided, but slide rules must not be used. They dropped the negative instruction about slide rules in1969 and this is the case in the years 1969-1977. From 1972 they added that a list of miscellaneous formula would be provided. In the Victorian Universities and Schools Examination Board handbook for 1978 in the materials for examinations section it is stated that slide rules and electronic calculators may be used for mathematics.

The General Mathematics and Mathematics A/B computer option[68]

A 'computing option' was available within the Victorian Year 12 General Mathematics course from the mid 1970's. The course gained popularity, and by 1979 was offered at many schools, though often students drew flowcharts without ever running the programs they represented, or ran iterative processes on calculators as they did not have adequate access to real computers.

The computing option of General Mathematics was transferred into Mathematics in the reforming of courses in 1985.

The course continued in Mathematics A as an option

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[67] Investigations Into the Introduction of Logarithm Tables in Victoria Chris Barling, Conference Proceedings 2003 MERGA

[68] The Victorian Universities and Schools Examination Board, Handbook of directions and prescriptions, August 1997

The progression in the use of calculators in Victorian Schools has been as follows[69]:

- 1978 scientific calculators
- 1997/8 approved graphics calculators permitted (examinations graphics calculator 'neutral')
- 1999 'assumed access' for graphics calculators in Mathematical Methods and Specialist Mathematics
- 2000 assumed access to graphics calculators for all VCE Mathematics Examinations
- 2002 Mathematical Methods (CAS) pilot study, 'assumed access' for approved CAS in pilot examinations.
- 2006-2009 assumed access to graphics calculator or CAS technology for both Further Mathematics examinations. Concurrent implementation of Mathematical Methods/Mathematical Methods (CAS) with common technology free Examination 1 and technology assumed for Examination 2 graphics calculator for Mathematical Methods, CAS for Mathematical Methods (CAS). Technology free Examination1 for Specialist Mathematics and technology assumed (graphics calculator or CAS) for Examination 2.
- 2010-2015 assumed access to graphics calculator or CAS technology for both Further Mathematics examinations. Technology free Examination 1 for Mathematical Methods (CAS) and Specialist Mathematics Examination 1. CAS technology assumed Examination 2 for for Mathematical Methods (CAS) and Specialist Mathematics.
- 2016-2020 . Assumed access to CAS technology for both Further Mathematics examinations. Technology free Examination 1 for Mathematical Methods and Specialist Mathematics Examination 1. CAS technology assumed Examination 2 for Mathematical Methods (CAS) and Specialist Mathematics.

Mathematical Methods (CAS) and Mathematica computer-based examination

The Victorian Curriculum and Assessment Authority conducted a three year trial, from 2011 to 2013, for computer-based delivery and student response to the extended answer section of VCE Mathematical Methods (CAS) Examination 2. The trial involved schools across government, Catholic and independent sectors and aimed to develop and support effective alignment between the use of technology in

[69] Evans, M., Leigh-Lancaster, D., Norton, P. (2003). The Victorian Curriculum and Assessment Authority (VCAA) Mathematical Methods Computer Algebra System (CAS) pilot study examinations 2003. Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, Geelong, VIC, 372-379

curriculum, pedagogy and assessment, using the computer algebra system software Mathematica, from Wolfram Research. A group of around 70 students from these schools sat the examination in 2013 where Mathematical Methods (CAS) Examination 2 was produced, delivered and sat as a computer-based exam, with the functionality of Mathematica as the computational tool. The VCAA has now expanded implementation of the Computer-Based Examination 2 for Mathematical Methods (CAS) to a further 5 schools, for CBE in 2016.

6 Concluding remarks

We first recall the concluding remarks of Professor Crossley [70] which were given in a paper for a previous review

> *Having determined what the context is in which mathematics is to be taught, and to whom, one needs to be aware of its human origins and also of its enormous potential for changing, and hopefully improving, the world. At the present time mathematics encompasses much more discrete mathematics than it ever did. This should be recognised and an informed choice be made between retaining what has traditionally been taught - which has tended to be continuous mathematics - and what is needed now - namely significantly more discrete mathematics and an adequate appreciation of structure. There is not time or space in school to retain everything presently in the syllabus and introduce more. That would also be unfair to both students and teachers. The best foundation is to treat a limited number of areas to a good depth and, at the same time, teach students how to learn mathematics so that they will later be able to venture into areas previously unknown to them. Technology should be used where appropriate, but both its benefits and its limitations should be made clear. The rôles of both proof and computation should be clearly presented and permeate all the work. Ideas of both proof and (mathematical) structure are fundamental and support each other. This should be properly recognised. Without the idea of structure, the idea of proof has no foundation. Also the usefulness of the language of mathematics for describing the world, even without proof or calculation, should be established. Finally a balance should be struck - and this is not easy to achieve - between, on the one hand, supporting, training and developing*

[70] The Essentials of, and for, Mathematics, John N. Crossley, Gazette of the Australian Mathematical Society,Volume 33 Number 3, July 2006.

> *those students who will be our successors as teachers and professional mathematicians and, on the other, trying to give some idea, however minimal, of mathematics, its usefulness, power and beauty to everyone.*

It does seem timely to carefully consider change in senior secondary mathematics. We know that the subject is useful to society in so many ways and none of the critics of the present state of mathematical education argue against this. In the current review some changes have been made that take us one step in the direction advocated by people such as Conrad Wolfram or Charles Fadel. A gentle evolution is necessary. The subject itself is has changed greatly in the past century and this is not reflected in our courses. The fact that our high level courses today are very similar to those introduced in 1967, nearly 50 years ago, should bring us to question what is in our courses. The world is not at all the same as it was in 1967,(two years after I matriculated). Should our courses have changed more than they have? This is an opportune time to introduce some new ideas into our curriculum and to start to plan and consult on changes for the future.