**Philip Swedosh -** Welcome to a video on-demand for Specialist Mathematics. There have been a number of changes to the Specialist Mathematics curriculum. One of the largest changes is the introduction of proofs. My name is Philip Swedosh, and in these videos, I will outline the structure required in questions which ask for a proof to be performed. Some of this may feel unfamiliar at first but once the logic and structure are understood most questions are relatively straightforward. In the first video, I'll go through proof by induction questions. In the second, we'll meet proof by contradiction.

To help with the understanding of proof by induction I will use an analogy with dominoes. Now, I believe we would've all seen a set of dominoes set up in a room where when you flick one domino it causes other dominoes to fall, and very often they're done in a way that makes a lovely pattern. Proof by induction is very similar. Basically, there's a template to this. What we need to do is we need to prove that the first domino will fall. Now, that might be n equals one quite often but it may be some other domino, so we prove that a domino falls. Then what we do is we assume that the k-th domino will fall. So we assume whatever we're trying to prove is true for n equals k. Now the third and most important step is that we need to prove that what we are doing is true for n equals k plus one. And we must use the assumption.

Now, the analogy here is we are showing that if the k-th domino falls that causes the k plus one-th domino to fall. In other words, if any arbitrary domino gets knocked over that causes the following one to be knocked over. And what that does is sets up a situation where any domino causes subsequent ones to fall, and therefore it will be true that all of the dominoes must fall. Now in the mathematical context, what we are doing is we are proving something is true for a particular number, a starting point, which we often label n-zero. We assume that the statement is true for n equals k and then we must use that assumption that the n equals k statement is true to prove that the statement for k plus one is then true.

We start off with a relatively straightforward example which contains an equality, so we need to prove the statement on the screen by induction for all n element of natural numbers. As we need to prove the statement for all natural numbers. We start by proving that it's true for n equals one. Now you'll notice in the working that in proof by induction we try to keep, we do keep the left and the right side quite separate, so we substitute one into the left side, we get a half, we substitute one into the right side, and we also get a a half.

So we've proven that it works for a starting point that being n equals one. We then make the assumption that the statement will be true for n equals k. Now I find it easier to write out what I'm assuming because I will ultimately be wanting to use that in the next step. Now we need to prove that it's true, that the statement is true for n equals k plus one. Now again, I like to write out the statement that I'm trying to prove, and if you look carefully, you'll see that the k plus one-th term is added in to the left hand side. So when we start our working, the left hand side is equal to one minus one over two to the power of k, which replaces all of the terms except for the last one, plus that last term which is included in this k plus one-th term.

We then put the middle term on a common denominator so effectively we're multiplying top and bottom by two. We then simplify and we show that it equals the right hand side. Hence, we have proven the original statement is true for all n. Now we're going to look at a proof by induction using an inequality. So this has a few little extra bits to it to the equality but we'll work through that as we go. So in part a, what we're going to do is we're going to consider that inequality and it's true for n is greater than equal to n-zero, which is our starting point.

The first part of the question asks us to show that n-zero is equal to five, and in the second part we'll prove that the statement is true for all n greater than equal to five where n is a natural number. To show that n-zero equals five. You can set this out a number of different ways, I've chosen to set it up in a table which is quite convenient. So I've used n-values from one to six and simply observed when two to the n becomes greater than n squared. And so we've got n-zero equals five. Now proving the statement is true for all n greater than equal to five, we've already done the first part. We've proven that it's true for n equals five. So that's done. We now make the assumption that the statement is true for n equals k. And again, I've written out what that would mean in terms of k.

We now need to prove that the statement is true for n equals k plus one using our assumption. Again, I've written down what the statement is, with n equals k plus one because it will just make our life easier later on. Now, the first thing I've done is I've expanded the right hand side. Note that I am still operating on each of the sides independently of the other, not both at once. So I've expanded the right hand side and you'll see on the next slide why this is very convenient. In the first line, we've simply used our index laws that two to the power of k plus one is the same as two times two to the power of k. Now using our assumption that must be greater than two times k squared. Now, this is where you have to always consider what you're trying to prove.

The right hand side is a quadratic starting with a k squared. So, we write the two k squared as k squared plus k squared so that we've matched the first term of that quadratic. We can then say that because k is greater than or equal to five, the previous line k squared plus k squared must be greater than or equal to k squared plus five k. In other words, we're replacing one of the k's and saying that it is five or more. Again to match the right hand side, we then splitting the 5k up into a 2k plus a 3k so that the first two terms now match the right hand side so it equals k squared plus two k plus three k. We then can again use the fact that k is at least five and say that three k therefore is at least 15.

So the previous quadratic is now greater than or equal to k squared plus two k plus 15. Clearly this is greater than k squared plus two k plus one which is the right hand side. So this uses a cascading type effect, if sum, if a is greater than b and b is equal to c but c is greater than d, it follows that a must be greater than d, and so we have proven it for all n greater than or equal to five. In this example, we'll be looking at the divisibility. We need to prove that the number nine to the power n minus five to the power n is divisible by four for all n, which are an element of natural numbers. As we've done previously, we first prove it for n equals one.

The left hand side then will equal nine minus five which is four. Now, clearly that's divisible by four so we don't need to go any further. We then want to make an assumption that the statement is true for n equals k. Now you can state this in a number of ways. What I've chosen to do here is to say that n to the power k minus five to the power k must equal four times a where a is a natural number. You could just say that it is divisible by four but I think this is more convenient for the rest of the setting out. Now, we wish to prove that it's true for n equals k plus one using the assumption that it's true for n equals k. Firstly, as I've done before, I've written out the statement in terms of the k plus one. So nine to the power of k plus one minus five to the power of k plus one is equal to four b, where b is a natural number so therefore it must be divisible by four. In the first line of working, I've written out just using the index laws nine times nine to the k minus five times five to the k.

Now by grouping a little we split it and factorise in order to use our assumption. So we've got five times nine to the power k minus five to the power k where the bit in brackets is part of our assumption and leftover is plus another four times nine to the k. We then use our assumption, in other words replace the nine to the k minus five to the k with four a. Then we're able to see that there's a common factor of four which we can factorise out of that. And if a is an integer, then five to the a plus nine to the k must be an integer, which we can call b and that is equal to our right hand side and therefore we have proven the statement and that it's divisible by four for all natural numbers.

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