VCE Specialist Mathematics
Units 3 and 4

Sample application task – Farey sequences

The Farey sequence *Fn*is the set of all irreducible rational numbers in the unit interval, [0, 1] with

denominator *b* less than or equal to *n*, arranged in increasing order. For example,

*F*5 = [ , , , , , , , , , , ]

(Irreducible means that the fraction cannot be simplified.)

You will find suitable references to Farey sequences on the internet.

1. Describe a simple algorithm in pseudocode to construct a function *Fn* by inspecting all the fractions with 1 ≤ *i* ≤ *j* and 1 ≤ *j* ≤ *n*; keeping only those where *i* and *j* have no common factor except 1; and then sorting the list. Use nested loops.
2. There is a better algorithm which we describe here:

 *Fn*= [ , , … , ]

1. Let *a*0 = 0, *b*0 = 0, *a*1= 1 and *b*1 = *n*
2. Define the following recursive formulas:
3. *ai*+1 = *mi**ai* – *ai*–1
4. *bi*+1 = *mi**bi* – *bi*–1
5. *mi* = integer part of
6. If *ai*+1 = 1 and *bi*+1 = 1 you are done.

The integer part of a number is the part before the decimal point. There are integer part functions built into most programming languages.

Work through the algorithm for a few steps to see how it works. You can use lists to obtain a neater print.

1. Write an algorithm which gives the number of terms in *Fn*.

One implementation for Question 2 in Python is shown:

import math

def farey(*n*):

 *a* = [0,1]

 *b* = [1,*n*]

 *nmax* = math.factorial(*n*)

 for *k* in range(1, *nmax*):

 *m* = math.floor((*b*[*k*-1] + *n*)/*b*[*k*])

 *a*.append(*m*\**a*[*k*] – *a*[*k*-1])

 *b*.append(*m*\**b*[*k*] - *b*[*k*-1])

 if (*a*[*k*+1] == 1) and (*b*[*k*+1] == 1):

 break

 return *a*, *b*

Why use *n* to define the loop which is too big. Think of other ways to terminate the program.

An output for farey(7) is

([0, 1, 1, 1, 1, 2, 1, 2, 3, 1, 4, 3, 2, 5, 3, 4, 5, 6, 1],

[1, 7, 6, 5, 4, 7, 3, 5, 7, 2, 7, 5, 3, 7, 4, 5, 6, 7, 1])

1. Obtain a proof for the following result. Induction can be used for the converse.

Given 0 ≤ ≤ ≤ 1, then and are Farey neighbours in *F*­*n* if and only if *bc* – *ad* = 1.

1. **Mediants**

For the two rational numbers  and  the mediant is defined as . Investigate the relation between the mediant of two fractions and their role in Farey sequences.

1. **The Totient function**

Euler’s Totient function ϕ(*n*) is the count of numbers in {1, 2, 3, …, *n*} that are relatively prime, up to *n*, i.e. the numbers whose HCF with *n* is 1.

This function gives a way of finding the number of terms in *F*­*n.* Write a program to form a function that will give the values of ϕ(*n*) for *n* = 1 to 50.

Investigate how the totient function may be used to find the number of terms in *F*­*n* through a recursion equation.

1. There are many other properties of Farey sequences to investigate. For example, their connection with the Fibonacci sequence.

Areas of study

The following content from the areas of study is addressed through this task.

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| **Area of study** | **Topic** | **Content dot point** |
| Discrete mathematics | Logic and proof | 1, 2, 3, 4, 5 |

Outcomes

The following outcomes, key knowledge and key skills are addressed through this task.

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| **Outcome** | **Key knowledge dot point** | **Key skill dot point** |
| **1** | 1, 5, 6, 7 | 1, 5 |
| **2** | 1, 2, 3, 4 | 1, 2, 4, 5, 7 |
| **3** | 1, 2, 3, 4, 5, 6 | 1, 2, 3, 4, 5, 6, 8, 9, 10, 11 |

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