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PROCESSING LABEL HERE

Write your **student number** in the boxes above. **Letter**

Specialist Mathematics
Examination 1

Question and Answer Book

VCE Examination – Monday 11 November 2024

- Reading time is **15 minutes**: 9.00 am to 9.15 am
- Writing time is **1 hour**: 9.15 am to 10.15 am

Materials supplied

- Question and Answer Book of 16 pages
- Formula Sheet

Students are **not** permitted to bring any technology (calculators or software) or notes of any kind into the examination room.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents pages

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Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.
- Unless otherwise specified, an **exact** answer is required for each question.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have a magnitude g m s^{-2}, where $g = 9.8$

Question 1 (4 marks)

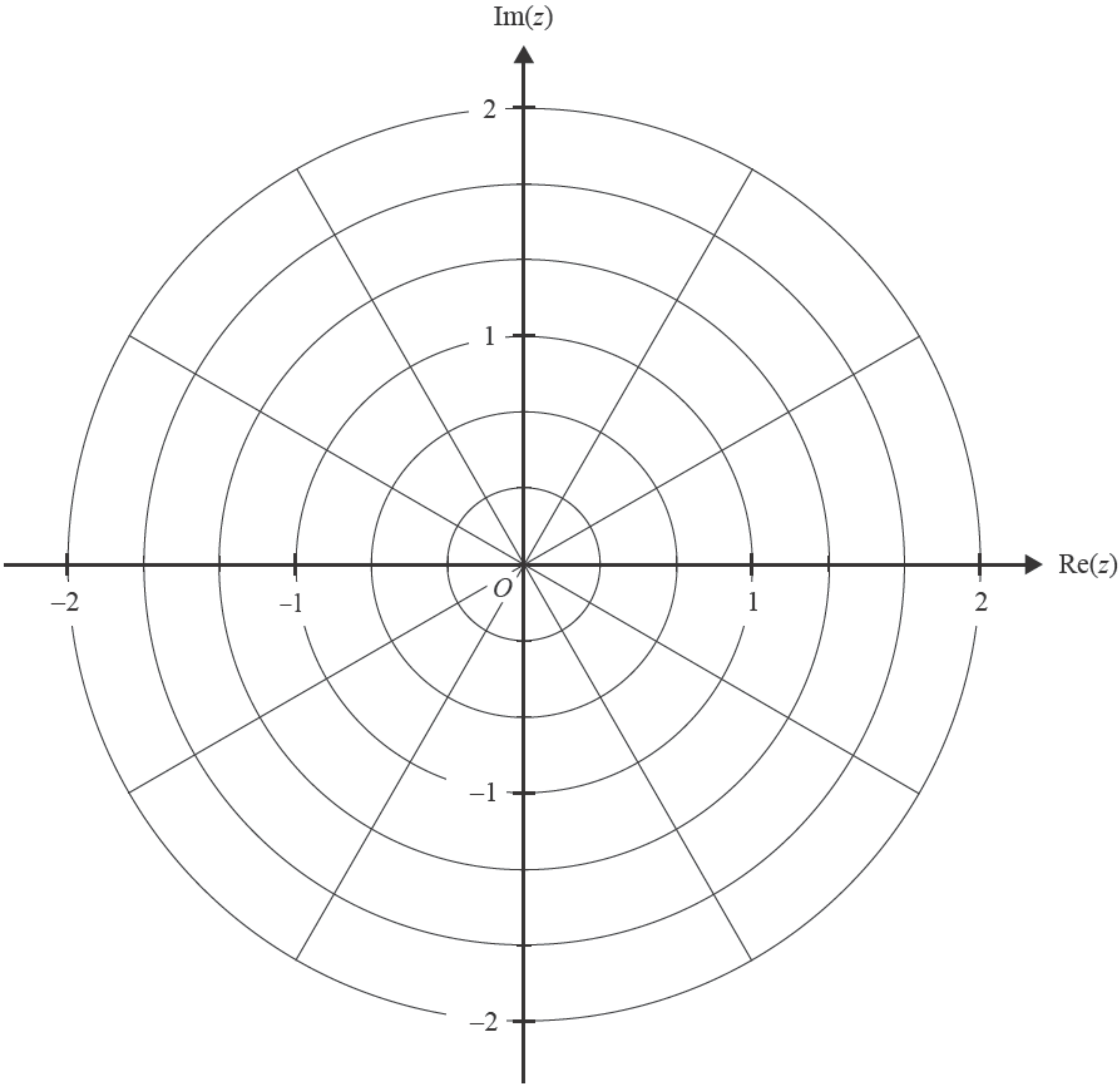
Consider the function with rule $f(z) = 3z^3 + 2iz^2 + 3z + 2i$, where $z \in C$.

a. Verify that $3z + 2i$ is a factor of $f(z)$. 1 mark

b. Hence or otherwise, solve the equation $f(z) = 0$.

Give your answers in Cartesian form. 2 marks

c. Plot the solutions of $f(z) = 0$ on the Argand diagram below. 1 mark



Question 2 (3 marks)

Prove that if x is an odd integer then $2x^2 - 3x - 7$ is even, using a direct proof.

Question 3 (6 marks)

Let $f: R\setminus\{-1\} \to R,\ f(x) = \dfrac{(x-1)^2}{(x+1)^2}$.

The rule $f(x)$ can be written in the form $f(x) = A + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$, where $A, B, C \in Z$.

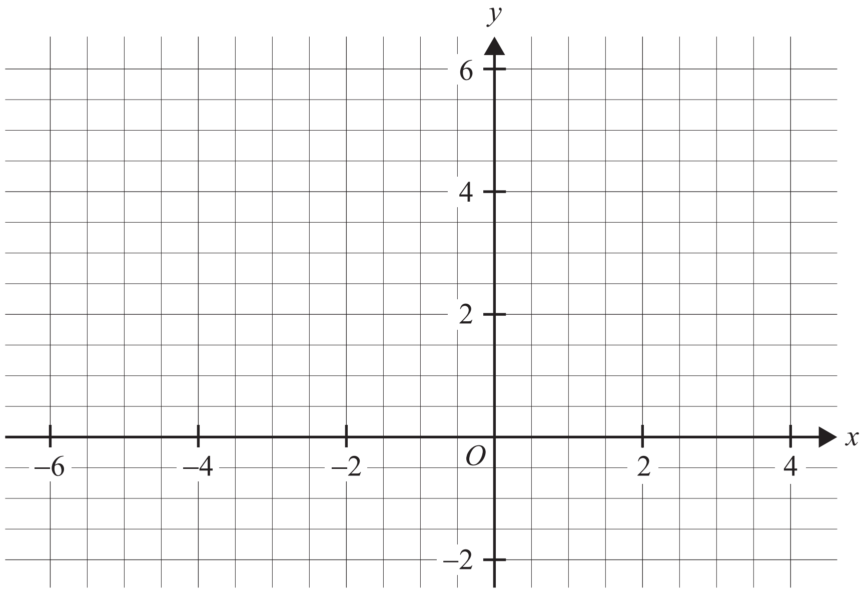
a. Show that $A = 1$, $B = -4$ and $C = 4$. 1 mark

b. The graph of f has one turning point.

Find the coordinates of this turning point. 2 marks

Do not write in this area.

c. Sketch the graph of $y = f(x)$ on the set of axes below. Label the asymptotes with their equations and the axial intercepts with their coordinates. 3 marks



Question 4 (4 marks)

Consider the vectors $\underset{\sim}{a} = 3\underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{c} = n\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, where $n \in Z$.

a. Find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. 2 marks

b. Find all possible values of n such that the dot product of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ is equal to the magnitude of the cross product of $\underset{\sim}{a}$ and $\underset{\sim}{c}$. 2 marks

Question 5 (3 marks)

The curve with equation $y = \sqrt{k - \frac{1}{x^2}}$, for $1 \le x \le \frac{k}{2}$ where $k > 2$, is rotated about the x-axis to form a solid of revolution that has volume $\frac{7\pi}{2}$ units^3.

Show that k satisfies the equation $k^3 - 2k^2 - 9k + 4 = 0$.

Question 6 (5 marks)

The production of a brand of weed trimmer involves three stages, Stage 1, Stage 2 and Stage 3, which take W_1 hours, W_2 hours and W_3 hours, respectively. Here W_1, W_2 and W_3 are independent random variables, which may be assumed to be normally distributed. Assume that Stage 2 starts immediately after Stage 1 ends and that Stage 3 starts immediately after Stage 2 ends.

The mean, standard deviation and cost at each stage are shown in the table below.

Stage	Time (h)	Mean (h)	Standard deviation (h)	Cost ($/h)
1	W_1	1.0	0.3	10
2	W_2	1.5	0.4	20
3	W_3	2.0	0.5	15

a. Find the mean and the variance of the total time to produce one weed trimmer. 1 mark

b. Find the variance of the total cost to produce one weed trimmer. 2 marks

Do not write in this area.

c. If a single weed trimmer is produced, find the probability that the time spent at Stage 2 will be less than the time spent at Stage 1.

Give your answer correct to two decimal places.

Use $\Pr(-1 < Z < 1) = 0.68$, where Z is the standard normal variable with mean 0 and standard deviation 1. 2 marks

Question 7 (4 marks)

Solve the differential equation $x + 2y\sqrt{x^2+1}\,\frac{dy}{dx} = 0$, expressing y as a function of x, given that $y(0) = -2$.

Do not write in this area.

Question 8 (4 marks)

Consider the relation $x^2 y^2 + xy = 2$, where $x, y \in R$.

a. Using implicit differentiation, show that $\frac{dy}{dx} = -\frac{y}{x}$ given that $2xy \neq -1$. 2 marks

b. Find all points on the graph of $x^2 y^2 + xy = 2$ where the slope of the tangent is equal to -1. 2 marks

Question 9 (4 marks)

A car is travelling along a straight, flat road. The velocity, v km h^{-1}, of the car and its position, x kilometres, are measured from the position on the road where $x = 0$.

The velocity v and the position x of the car are related by $v^2 = 1600 + \frac{672}{\pi}\arccos\left(\frac{x}{20}\right)$, where $-15 \le x \le 15$ and $v \ge 0$.

A speed detection device is positioned to detect the speed of a car as it passes the position $x = 0$. The speed limit on the road is 40 km h^{-1}.

The speed detection device will be activated if the car is travelling at 10% or more above the speed limit.

a. Determine, with evidence, whether the speed detection device will be activated. 1 mark

b. Find the acceleration of the car, in km h^{-2}, when $x = 12$.
Give your answer in the form $\frac{k}{\pi}$, where $k \in Z$. 3 marks

Question 10 (3 marks)

Let the lines l_1 and l_2 be defined by

$l_1 : \underset{\sim}{r}_1(\lambda) = \underset{\sim}{i} + m\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$ and $l_2 : \underset{\sim}{r}_2(\mu) = 2\underset{\sim}{i} - \underset{\sim}{k} + \mu\left(-\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)$, where

$m \in R \setminus \left\{-\frac{4}{5}\right\}$ and $\lambda, \mu \in R$.

If the shortest distance between the two skew lines l_1 and l_2 is $\frac{14}{\sqrt{35}}$, find the values of m.

End of examination questions

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Specialist Mathematics Examination 1

2024 Formula Sheet

You may keep this Formula Sheet.

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Mensuration

area of a circle segment	$\frac{r^2}{2}\left(\theta - \sin(\theta)\right)$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$	sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
volume of a pyramid	$\frac{1}{3}Ah$	cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r\left(\cos(\theta) + i\sin(\theta)\right) = r\text{cis}(\theta)$	$\lvert z \rvert = \sqrt{x^2 + y^2} = r$	
$-\pi < \text{Arg}(z) \le \pi$	$z_1 z_2 = r_1 r_2 \text{cis}\left(\theta_1 + \theta_2\right)$	
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}\left(\theta_1 - \theta_2\right)$	de Moivre's theorem	$z^n = r^n \text{cis}(n\theta)$

Data analysis, probability and statistics

<table>
<tr><td rowspan="2">for independent random variables $X_1, X_2, \ldots, X_n$</td><td colspan="2">$\mathrm{E}(aX_1 + b) = a\,\mathrm{E}(X_1) + b$
$\mathrm{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$
$= a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \ldots + a_n\mathrm{E}(X_n)$</td></tr>
<tr><td colspan="2">$\mathrm{Var}(aX_1 + b) = a^2\mathrm{Var}(X_1)$
$\mathrm{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$
$= a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \ldots + a_n^2\mathrm{Var}(X_n)$</td></tr>
<tr><td rowspan="2">for independent identically distributed variables $X_1, X_2, \ldots, X_n$</td><td colspan="2">$\mathrm{E}(X_1 + X_2 + \ldots + X_n) = n\mu$</td></tr>
<tr><td colspan="2">$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = n\sigma^2$</td></tr>
<tr><td>approximate confidence interval for μ</td><td colspan="2">$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$</td></tr>
<tr><td rowspan="2">distribution of sample mean $\bar{X}$</td><td>mean</td><td>$E(\bar{X}) = \mu$</td></tr>
<tr><td>variance</td><td>$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$</td></tr>
</table>

Calculus

$\frac{d}{dx}\left(x^n\right) = n x^{n-1}$	$\int x^n dx = \frac{1}{n+1} x^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = a e^{ax}$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e \lvert x \rvert + c$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$	$\int \sec^2(ax)\, dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}\left(\cot(ax)\right) = -a\,\mathrm{cosec}^2(ax)$	$\int \mathrm{cosec}^2(ax) dx = -\frac{1}{a}\cot(ax) + c$
$\frac{d}{dx}\left(\sec(ax)\right) = a\sec(ax)\tan(ax)$	$\int \sec(ax)\tan(ax)\, dx = \frac{1}{a}\sec(ax) + c$
$\frac{d}{dx}\left(\mathrm{cosec}(ax)\right) = -a\,\mathrm{cosec}(ax)\cot(ax)$	$\int \mathrm{cosec}(ax)\cot(ax)\, dx = -\frac{1}{a}\mathrm{cosec}(ax) + c$
$\frac{d}{dx}\left(\sin^{-1}(ax)\right) = \frac{a}{\sqrt{1-(ax)^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\cos^{-1}(ax)\right) = \frac{-a}{\sqrt{1-(ax)^2}}$	$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\tan^{-1}(ax)\right) = \frac{a}{1+(ax)^2}$	$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n\, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\quad n \neq -1$
	$\int \frac{1}{ax+b} dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$

Calculus – continued

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
integration by parts	$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
Euler's method	If $\frac{dy}{dx} = f(x, y)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f\left(x_n, y_n\right)$.
arc length parametric	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area Cartesian about x-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
surface area Cartesian about y-axis	$\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
surface area parametric about x-axis	$\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area parametric about y-axis	$\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	
constant acceleration formulas	$v = u + at$	$s = ut + \frac{1}{2}at^2$
	$v^2 = u^2 + 2as$	$s = \frac{1}{2}(u + v)t$

Vectors in two and three dimensions

$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
for $\underset{\sim}{r}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$	vector scalar product $\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = \lvert\underset{\sim}{r}_1\rvert\lvert\underset{\sim}{r}_2\rvert\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
	vector cross product $\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$
vector equation of a line	$\underset{\sim}{r}(t) = \underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_1 + x_2t)\underset{\sim}{i} + (y_1 + y_2t)\underset{\sim}{j} + (z_1 + z_2t)\underset{\sim}{k}$
parametric equation of a line	$x(t) = x_1 + x_2t \quad y(t) = y_1 + y_2t \quad z(t) = z_1 + z_2t$
vector equation of a plane	$\underset{\sim}{r}(s, t) = \underset{\sim}{r}_0 + s\underset{\sim}{r}_1 + t\underset{\sim}{r}_2$ $= (x_0 + x_1s + x_2t)\underset{\sim}{i} + (y_0 + y_1s + y_2t)\underset{\sim}{j} + (z_0 + z_1s + z_2t)\underset{\sim}{k}$
parametric equation of a plane	$x(s, t) = x_0 + x_1s + x_2t,\ y(s, t) = y_0 + y_1s + y_2t,\ z(s, t) = z_0 + z_1s + z_2t$
Cartesian equation of a plane	$ax + by + cz = d$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \text{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$
$\sin^2(ax) = \dfrac{1}{2}\left(1-\cos(2ax)\right)$	$\cos^2(ax) = \dfrac{1}{2}\left(1+\cos(2ax)\right)$

End of Formula Sheet