

VCE Specialist Mathematics 2023-2027 Units 3&4

Suggested approaches to
implementing the 2023-2027
study design

Calculus

Acknowledgement of Country

The VCAA respectfully acknowledges the Traditional Owners of Country throughout Victoria and pays respect to the ongoing living cultures of First Peoples.



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VCE Mathematics On-demand Videos

To support the implementation of the 2023-2027 study design for Mathematics, we have developed a series of short on-demand videos outlining approaches that teachers may wish to utilise in the classroom.

The information presented in these on-demand videos has been developed by current VCE teachers, in conjunction with the VCAA, and offer suggestions for ways schools could approach the implementation of the 2023-2027 VCE Mathematics study design.

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General Outline

1. Integration by parts
2. Surface Area of Revolution
3. Logistic Equation



1. Integration by parts

The method of **integration by parts** comes from the product rule of differentiation. It allows us to integrate a function which is written as a product.

The product rule is:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Integrate both sides with respect to x :

$$\int \frac{d}{dx}(uv) dx = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$



Integration by parts Formula

Example:

Find $\int x^2 \cos(2x) dx$

Solution

Let $u = x^2$ and $\frac{dv}{dx} = \cos(2x)$.

$\frac{du}{dx} = 2x$ and $v = \frac{1}{2} \sin(2x)$



$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\Rightarrow \int x^2 \cos(2x) dx = \int u \frac{dv}{dx} dx$$

$$= uv - \int v \frac{du}{dx} dx$$

$$= x^2 \times \frac{1}{2} \sin(2x) - \int 2x \times \frac{1}{2} \sin(2x) dx$$

$$= \frac{1}{2} x^2 \sin(2x) - \int x \sin(2x) dx$$



Solution continue

$$u = x \quad \text{and} \quad \frac{dv}{dx} = \sin(2x).$$

$$\frac{du}{dx} = 1 \quad \text{and} \quad v = -\frac{1}{2} \cos(2x)$$

$$\begin{aligned} \int x^2 \cos(2x) dx &= \frac{1}{2} x^2 \sin(2x) - \left(x \times -\frac{1}{2} \cos(2x) - \int 1 \times -\frac{1}{2} \cos(2x) dx \right) \\ &= \frac{1}{2} x^2 \sin(2x) + \frac{1}{2} x \cos(2x) - \frac{1}{2} \int \cos(2x) dx \\ &= \frac{1}{2} x^2 \sin(2x) + \frac{1}{2} x \cos(2x) - \frac{1}{4} \sin(2x) + c \end{aligned}$$



2. Area of Revolution

Rotation about the x -axis

If the curve $y = f(x)$ from $x = a$ to $x = b$ is rotated about the x -axis, then the area of the surface of revolution is given by

$$A = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Example

The curve given by $y = \sqrt{4 - x^2}$, where $x \in [-1, 1]$ is rotated about the x -axis to form a solid of revolution.

Find the total surface area of the **closed** volume of this solid.



Solution

$$y = (4 - x^2)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{4 - x^2}}$$

$$A = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\begin{aligned} A &= 2\pi \int_{-1}^1 \sqrt{4 - x^2} \sqrt{1 + \frac{x^2}{4 - x^2}} dx \\ &= 2\pi \int_{-1}^1 \sqrt{4 - x^2} \sqrt{\frac{4}{4 - x^2}} dx \end{aligned}$$

$$\begin{aligned} A &= 2\pi \int_{-1}^1 2 dx \\ &= 4\pi [x]_{-1}^1 \\ &= 8\pi \end{aligned}$$

The total surface area of the **closed** volume of this solid is

$$\begin{aligned} &8\pi + 2 \times \left(\pi \times \sqrt{3}^2 \right) \\ &= 14\pi \text{ units}^2 \end{aligned}$$



2. Area of Revolution continue

Rotation about the y -axis

If the curve $y = f(x)$ from $y = a$ to $y = b$ is rotated about the y -axis, then the area of the surface of revolution is given by

$$A = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Example

The curve given by $y = \sqrt[3]{x}$ is rotated about the y -axis to form a solid of revolution. Find the surface area of the part of this solid where $x \in [0, 8]$.



Solution

$$x = y^3 \Rightarrow \frac{dx}{dy} = 3y^2, \quad y \in [0, 2]$$

$$A = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$A = 2\pi \int_0^2 y^3 \sqrt{1 + (3y^2)^2} dy$$

$$= 2\pi \int_0^2 y^3 \sqrt{1 + 9y^4} dy$$

$$\text{Let } u = 1 + 9y^4 \Rightarrow \frac{du}{dy} = 36y^3$$

y	u
0	$1 + 9(0) = 1$
2	$1 + 9(2^4) = 145$

$$A = 2\pi \int_1^{145} y^3 u^{\frac{1}{2}} \frac{du}{36y^3}$$

$$= \frac{\pi}{18} \int_1^{145} u^{\frac{1}{2}} du$$

$$= \frac{\pi}{27} \left[u^{\frac{3}{2}} \right]_1^{145}$$

$$= \frac{\pi}{27} (145\sqrt{145} - 1) \text{ units}^2$$



2. Area of Revolution continue

Surface area using parameters

Consider a curve defined by the parametric equation $x = f(t)$ and $y = g(t)$ for $a \leq t \leq b$. The surface area of revolution formed by rotating the curve about the x -axis is given by

$$A = 2\pi \int_a^b |g(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example:

Find the surface area of revolution formed when the curve defined by

$$x = \frac{4}{3}\sqrt{(t+1)^3}, \quad y = \frac{1}{2}t^2 \quad \text{and} \quad 0 \leq t \leq 1 \quad \text{is rotated about the } x\text{-axis.}$$



Solution

$$x = \frac{4}{3}(t+1)^{\frac{3}{2}} \Rightarrow \frac{dx}{dt} = 2(t+1)^{\frac{1}{2}}$$

$$y = \frac{1}{2}t^2 \Rightarrow \frac{dy}{dt} = t$$

$$A = 2\pi \int_a^b |g(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$A = 2\pi \int_0^1 \frac{1}{2} t^2 \sqrt{4(t+1) + t^2} dt$$

$$= \pi \int_0^1 t^2 (t+2) dt$$

$$= \pi \int_0^1 t^3 + 2t^2 dt$$

$$= \pi \left[\frac{t^4}{4} + \frac{2t^3}{3} \right]_0^1$$

$$= \frac{11\pi}{12} \text{ units}^2$$



3. Logistic Equation

Example:

The population $P(t)$ of bacteria in a Petri dish satisfies the logistic differential equation

$$\frac{dP}{dt} = 2P \left(6 - \frac{P}{8000} \right)$$

where t is measured in hours and the initial population is 4000 bacteria.

- Find the maximum number of bacteria predicted by this model.
- Find the number of bacteria when the population is growing the fastest.
- Solve the differential equation to find P as a function of t .



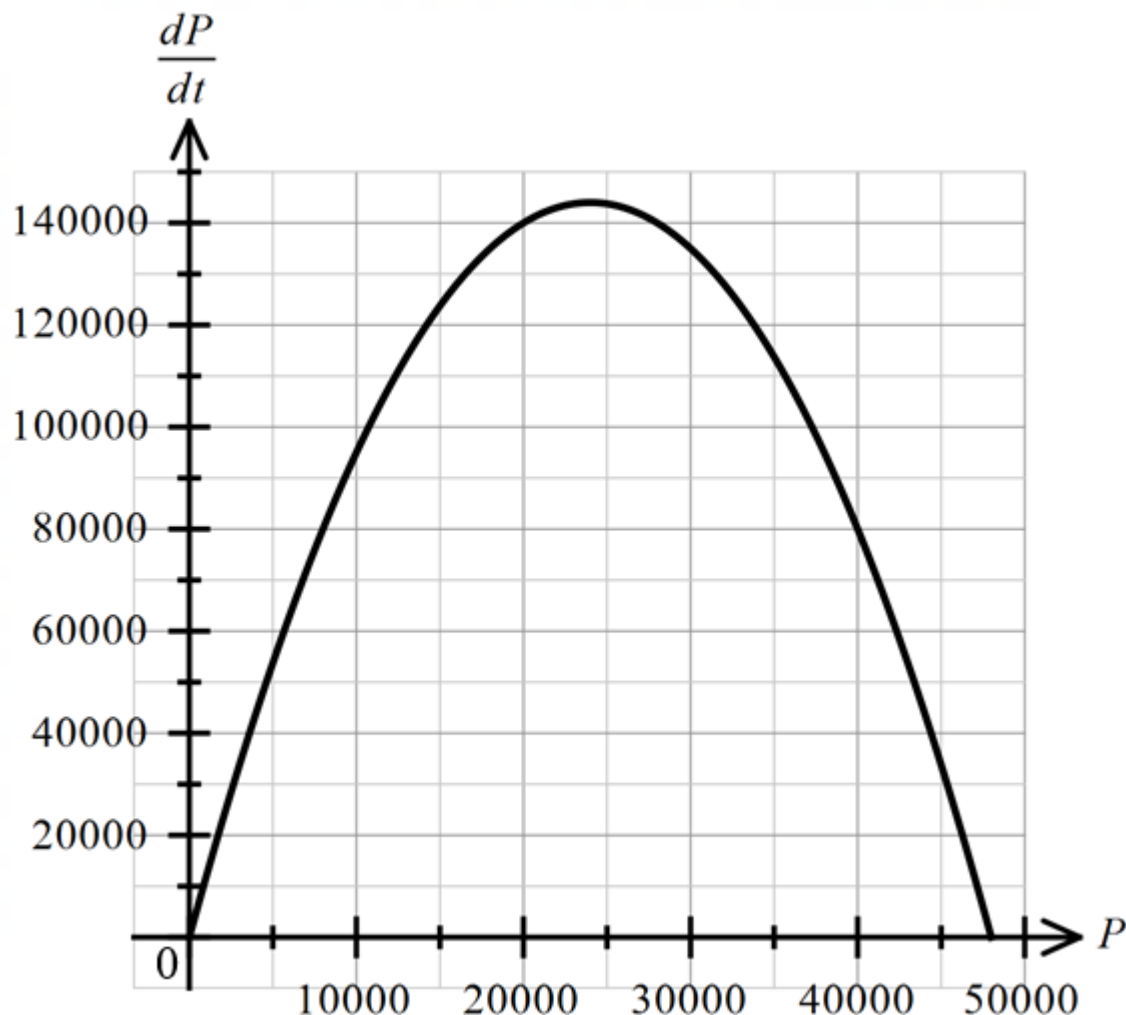
Solutions

a. Maximum when $\frac{dP}{dt} = 0$

$$\Rightarrow 2P \left(6 - \frac{P}{8000} \right) = 0$$

$$\Rightarrow P = 0 \quad \text{or} \quad 6 - \frac{P}{8000} = 0$$

$\therefore P = 48\,000$ bacteria is the maximum

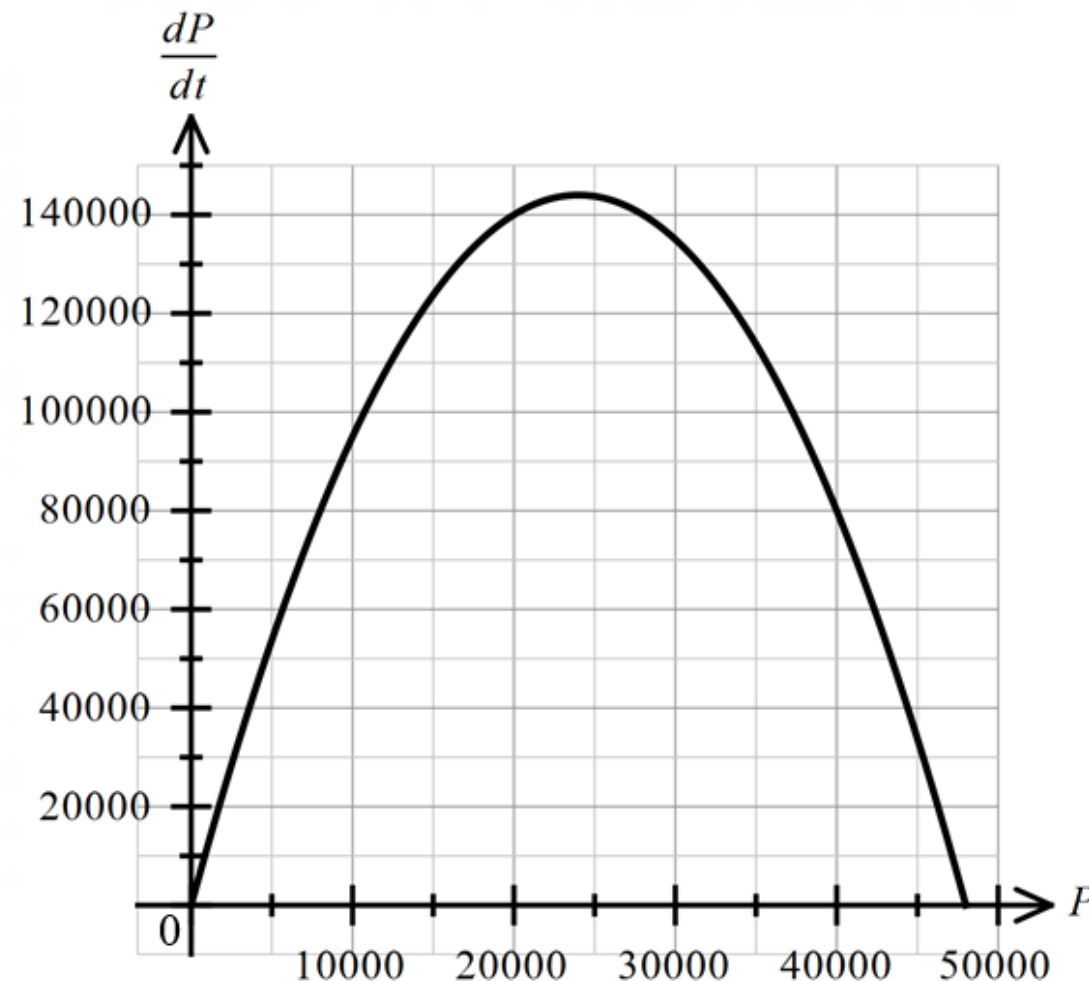


Solutions

- b.** The population is growing the fastest
ie the maximum of the derivative function.

$$P = \frac{0 + 48000}{2} = 24000$$

Therefore the population is growing fastest
when $P = 24000$ bacteria



Solutions

$$\text{c. } \frac{dt}{dP} = \frac{4000}{P(48000 - P)}$$

$$t = \frac{1}{12} \int_{4000}^P \left(\frac{1}{u} + \frac{1}{48000 - u} \right) du$$

$$= \frac{1}{12} \left[\ln(|u|) - \ln(|48000 - u|) \right] \Big|_{4000}^P$$

$$= \frac{1}{12} \left(\ln \left(\left| \frac{P}{48000 - P} \right| \right) - \ln \left(\frac{4000}{44000} \right) \right)$$

$$= \frac{1}{12} \ln \left(\frac{11P}{48000 - P} \right)$$

$$\therefore P = \frac{48000}{1 + 11e^{-12t}} \quad \text{bacteria}$$



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