

VCE Specialist Mathematics 2023-2027 Units 3&4

Suggested approaches to
implementing the 2023-2027
study design

Vector technique

Acknowledgement of Country

The VCAA respectfully acknowledges the Traditional Owners of Country throughout Victoria and pays respect to the ongoing living cultures of First Peoples.



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VCE Mathematics On-demand Videos

To support the implementation of the 2023-2027 study design for Mathematics, we have developed a series of short on-demand videos outlining approaches that teachers may wish to utilise in the classroom.

The information presented in these on-demand videos has been developed by current VCE teachers, in conjunction with the VCAA, and offer suggestions for ways schools could approach the implementation of the 2023-2027 VCE Mathematics study design.

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General Outline

1. Vector product
2. Planes in three dimensions
3. Angle between the plane and the line



1. Vector product

The **vector product** (also known as **vector cross product**) arises when we attempt to find a vector which is *perpendicular* to two other known vectors.

Consider vectors $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

The **vector cross product** (3X3 determinant shown on the next slide)

$$\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

The *magnitude* of **vector cross product**

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin(\theta) \text{ , where } \theta \text{ is the angle between } \underline{a} \text{ and } \underline{b}$$

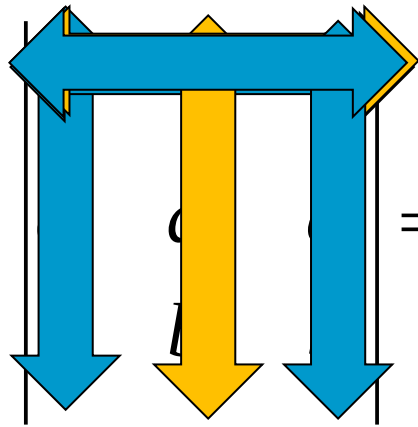


1. Vector product: co-factor expansion

The **vector cross product** can be found using co-factor expansion.

Consider vectors $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

The **vector cross product** is given by



$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \underline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \underline{k}$$

$$= (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

1. Vector product continued

Facts

$\underline{a} \times \underline{b} = \underline{0}$ then $\underline{a}$ and $\underline{b}$ will be parallel vectors

$\underline{a} \times \underline{b} \neq \underline{0}$ then $\underline{a} \times \underline{b}$ is orthogonal to both $\underline{a}$ and $\underline{b}$

Properties

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

$|\underline{a} \times \underline{b}|$ is the Area of a parallelogram, where $\underline{a}$ and $\underline{b}$ form two adjacent sides of a parallelogram.



2. Planes in three dimensions

Vector equation of a plane

$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$$

Where $\underline{r}$ is the position vector of any point on the plane,
 $\underline{a}$ is the position vector of a known point A on the plane, and
the vector $\underline{n} = n_1\underline{i} + n_2\underline{j} + n_3\underline{k}$ is normal to the plane

Cartesian equation of a plane

$$n_1x + n_2y + n_3z = D$$

Where n_1 , n_2 and n_3 are the components of the normal vector $\underline{n}$
and D is a constant.



Example 1

Find the Cartesian equation of the plane which is perpendicular to the vectors $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k}$ and $\underline{b} = 4\underline{i} + 2\underline{j} - 3\underline{k}$, which passes through the point (3, 2, 1).

Solution

The normal vector to the plane:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 1 \\ 4 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} \underline{i} - \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} \underline{j} + \begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix} \underline{k}$$
$$= (-3 \times -3 - 1 \times 2) \underline{i} - (2 \times -3 - 1 \times 4) \underline{j} + (2 \times 2 - -3 \times 4) \underline{k}$$
$$= 7\underline{i} + 10\underline{j} + 16\underline{k}$$



Solution continue

$\underline{a} \times \underline{b} = 7\underline{i} + 10\underline{j} + 16\underline{k}$, this is the normal vector to the plane.

That is, $\underline{n} = 7\underline{i} + 10\underline{j} + 16\underline{k}$

Cartesian equation of a plane $n_1x + n_2y + n_3z = D$

$$\Rightarrow 7x + 10y + 16z = D$$

Substitute the point (3, 2, 1) into the above equation:

$$7 \times 3 + 10 \times 2 + 16 \times 1 = D$$

$$\Rightarrow D = 57$$

$$\therefore 7x + 10y + 16z = 57$$



3. Angle between the plane and the line

Vector equation of a line

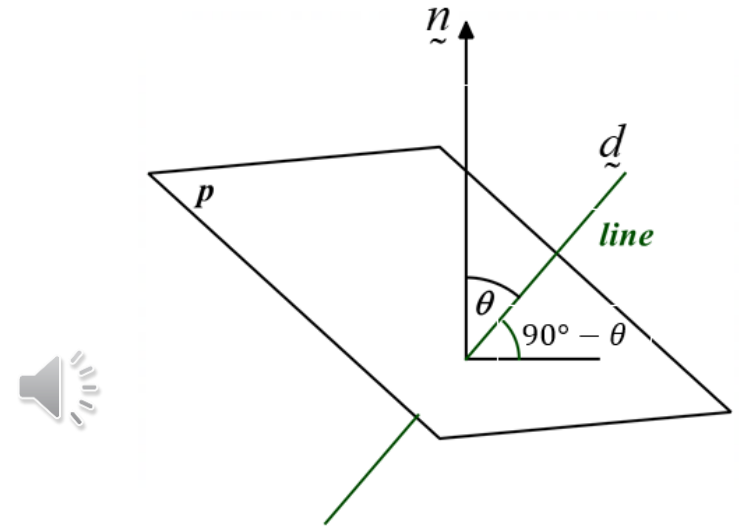
$$\underline{r} = \underline{a} + t\underline{d}, \quad t \in \mathbb{R}$$

Where $\underline{r}$ is the position vector at any point on the plane,
 $\underline{a}$ is the position vector of a point on the plane, and
 $\underline{d} = d_1\underline{i} + d_2\underline{j} + d_3\underline{k}$ is the direction vector of the line

Dot product: $\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 \cos(\theta)$

Where θ is the angle between two vectors.

Angle between the plane and the line is $90^\circ - \theta$,
where θ is the acute angle between the line and
the normal vector to the plane.



Example 2

Find the angle between the plane $2x + y + z = 7$ and the line given by

$$\underline{r} = 11\underline{i} + 4\underline{j} + 3\underline{k} + t(\underline{i} + 2\underline{j} - \underline{k}), \text{ where } t \in R$$

Solution

Cartesian equation of a plane $n_1x + n_2y + n_3z = D$

Hence, the normal vector to the plane is $\underline{n} = 2\underline{i} + \underline{j} + \underline{k}$

Vector equation of a line $\underline{r} = \underline{a} + t\underline{d}, \quad t \in R$

Hence, the direction vector of the line is $\underline{d} = \underline{i} + 2\underline{j} - \underline{k}$



Solution continue

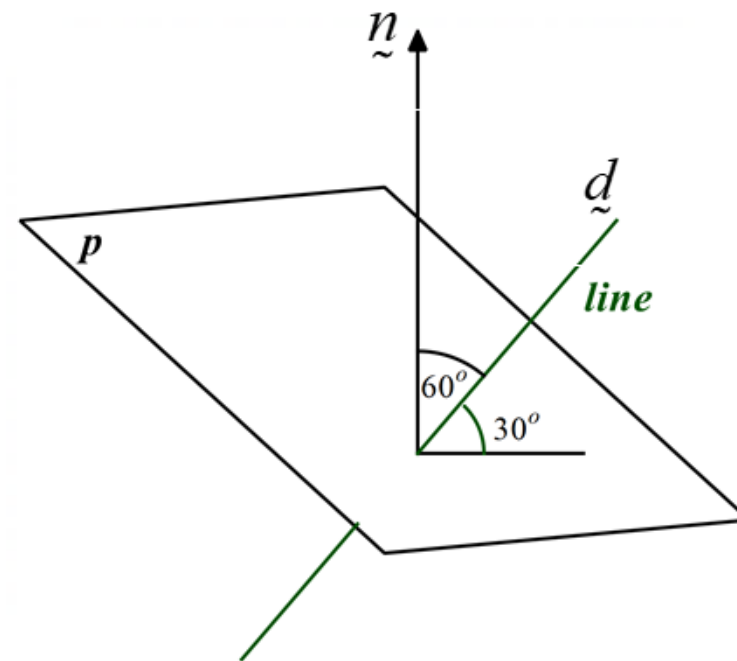
Let θ be the angle between the normal and the line

Dot product: $\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 \cos(\theta)$

$$\begin{aligned}\cos(\theta) &= \frac{(2\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{2^2 + 1^2 + 1^2} \times \sqrt{1^2 + 2^2 + 1^2}} \\ &= \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}\end{aligned}$$

Hence, the angle between the normal and the line is $\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$.

Therefore, the angle between the line and the plane is then $90^\circ - 60^\circ = 30^\circ$.



Example 3

a. Find the vector equation of the line through the points $A(3, 1, -1)$ and $B(5, 2, -6)$

Solution

Let $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = 5\underline{i} + 2\underline{j} - 6\underline{k}$

Direction vector, $\overrightarrow{AB}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underline{b} - \underline{a} = 2\underline{i} + \underline{j} - 5\underline{k}$$

Vector equation of a line $\underline{r} = \underline{a} + t\underline{d}$, $t \in R$

The equation of the line through point A with direction vector $\overrightarrow{AB}$ is

$$\underline{r} = 3\underline{i} + \underline{j} - \underline{k} + t(2\underline{i} + \underline{j} - 5\underline{k}), t \in R$$



b. Find the sine of the angle this line makes with the plane given by $x + 2y - z = 9$

Solution

The normal to the plane is $\underline{\underline{n}} = \underline{\underline{i}} + 2\underline{\underline{j}} - \underline{\underline{k}}$

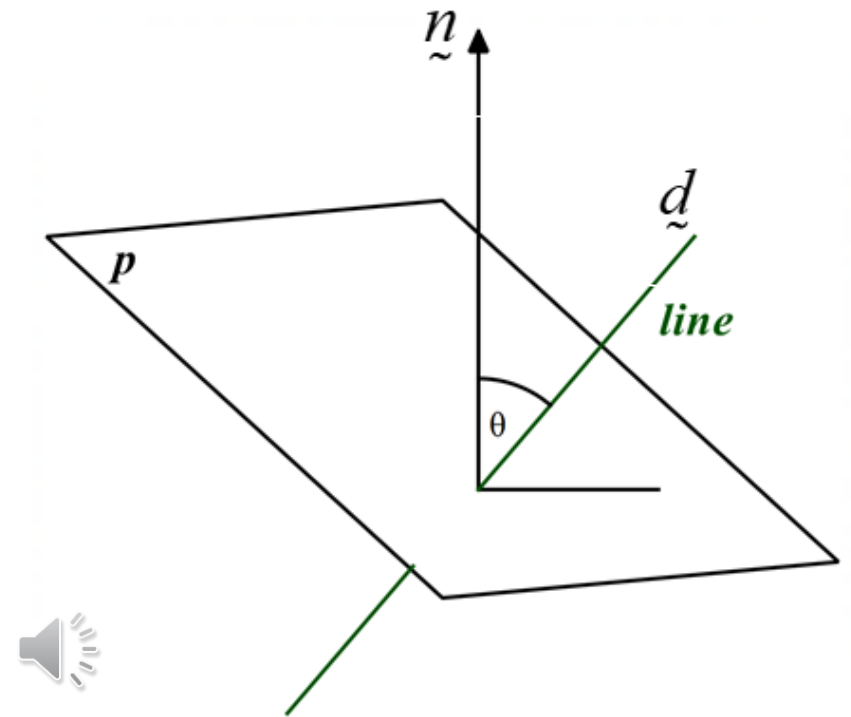
Let θ be the angle between the line and the normal vector.

$$\cos(\theta) = \frac{(2\underline{\underline{i}} + \underline{\underline{j}} - 5\underline{\underline{k}}) \cdot (\underline{\underline{i}} + 2\underline{\underline{j}} - \underline{\underline{k}})}{\sqrt{2^2 + 1^2 + 5^2} \times \sqrt{1^2 + 2^2 + 1^2}} = \frac{9}{\sqrt{30} \times \sqrt{6}} = \frac{3}{2\sqrt{5}}$$

The angle between the line and the plane is $\frac{\pi}{2} - \theta$

$$\Rightarrow \sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) = \frac{3}{2\sqrt{5}}$$

Therefore, the sine of the angle is $\frac{3}{2\sqrt{5}}$



Example 4

The position of a particle after t seconds is given by

$$\underline{r}(t) = t^2 \underline{i} + 5t \underline{j} + (t^2 - 16t) \underline{k}, \text{ where } t \geq 0$$

where components are measured in metres.

Find when the particle has minimum speed and find that minimum speed in ms^{-1} .

$$\underline{v}(t) = 2t \underline{i} + 5 \underline{j} + (2t - 16) \underline{k}$$

$$\begin{aligned} |\underline{v}(t)| &= \sqrt{4t^2 + 25 + (2t - 16)^2} \\ &= \sqrt{8t^2 - 64t + 281} \end{aligned}$$

The expression under the root sign is a positive quadratic, so the minimum is at its turning point.

Minimum speed is at $t = 4$, the minimum speed is

$$\sqrt{153} = 3\sqrt{17}$$



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