VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2021

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Letter

STUDENT NUMBER | | | | | | | | | |

MATHEMATICAL METHODS

Written examination 1

Friday 28 May 2021

Reading time: 2.00 pm to 2.15 pm (15 minutes)
Writing time: 2.15 pm to 3.15 pm (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 14 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given, unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (4 marks)

a. Find the derivative of $\dfrac{e^{2x}}{2x+1}$. 2 marks

b. Let $f: R \to R, f(x) = \sin^4(2x)$.

Evaluate $f'\left(\dfrac{\pi}{4}\right)$. 2 marks

Question 2 (4 marks)

Let $h : R^+ \rightarrow R$, $h(x) = x^3 \log_e(2x)$.

a. Show that $h'(x) = 3x^2 \log_e(2x) + x^2$. 1 mark

b. Let $\frac{dy}{dx} = 3x^2 \log_e(2x)$. The graph of y passes through the point $\left(\frac{1}{2}, -\frac{25}{24}\right)$.

Find the rule of y. 3 marks

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Question 3 (4 marks)

Steffi is raising money for her school with a lucky dip game.

In this game, 100 identical cubes, numbered 1 to 100, are placed in a large container. The container is then thoroughly shaken.

A player pays \$2 and is blindfolded so they cannot see. The player then selects a cube at random. If the number on the selected cube is a multiple of 10 (that is, 10, 20, 30, …, 100), the player wins a cash prize of \$7.

The cube is then returned to the container, which is thoroughly shaken again, before the next player makes a random selection. Each selection is independent of previous selections.

a. Find the probability that a player will win a cash prize of \$7. 1 mark

b. What is the expected loss to a player per game? 2 marks

c. $\hat{P}$ is the random variable that represents the proportion of games won by a player in random samples of three games played.

Find $\Pr\left(\hat{P} = \frac{2}{3}\right)$. 1 mark

Question 4 (5 marks)

Let $f: R \to R, f(x) = 2e^x + 1$ and let $g: (-2, \infty) \to R, g(x) = \log_e(x + 2)$.

a. **i.** Find $f\left(g(x)\right)$ in the form $ax + b$, where $a, b \in R$. 1 mark

ii. State the range of $f\left(g(x)\right)$. 1 mark

b. Let $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ and let the graph of the function h be the transformation of the graph of the function g under T.

If $h = f^{-1}$, then find the values of c and d. 3 marks

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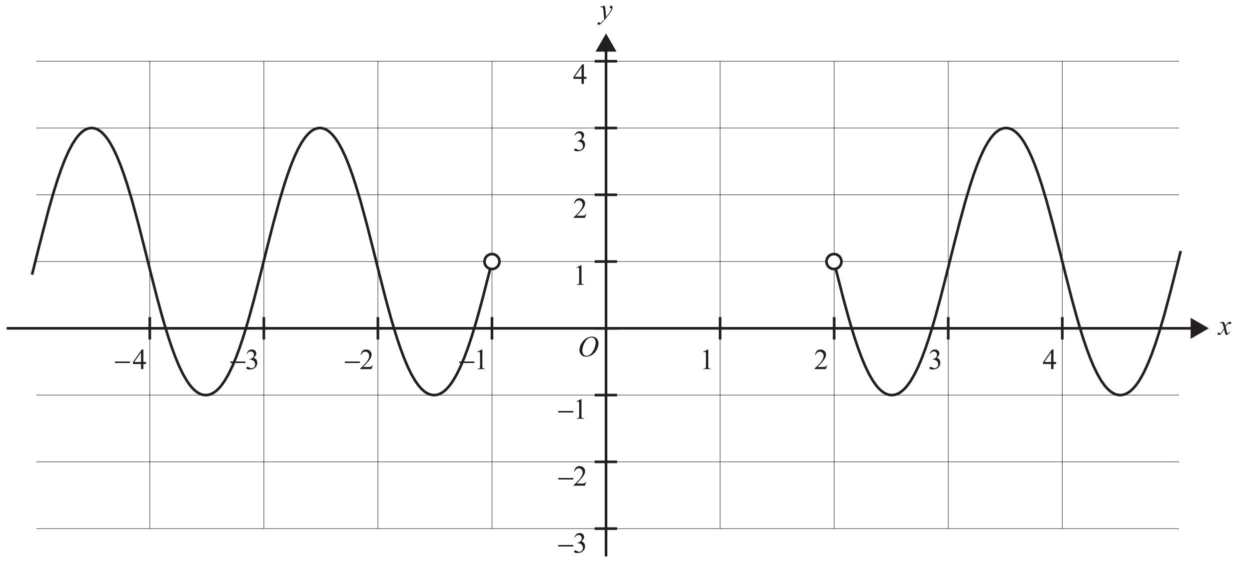
Question 5 (5 marks)

Part of the graph of $f : (-\infty, -1) \cup (2, \infty) \to R$, $f(x) = -2\sin(\pi x) + 1$ is shown below.



Let $g : [-1, 2] \to R$, $g(x) = -2\sin(\pi x) + 1$.

a. Sketch the graph of g on the axes provided above. 1 mark

b. Find the solutions to $g(x) = 0$. 2 marks

c. Find the area of the region bounded by the graph of g and the x-axis. 2 marks

Question 6 (3 marks)

Let A and B be events for a sample space such that $\Pr(A) = \frac{2}{3}$, $\Pr(B|A) = \frac{2}{5}$ and $\Pr(B|A') = \frac{1}{4}$.

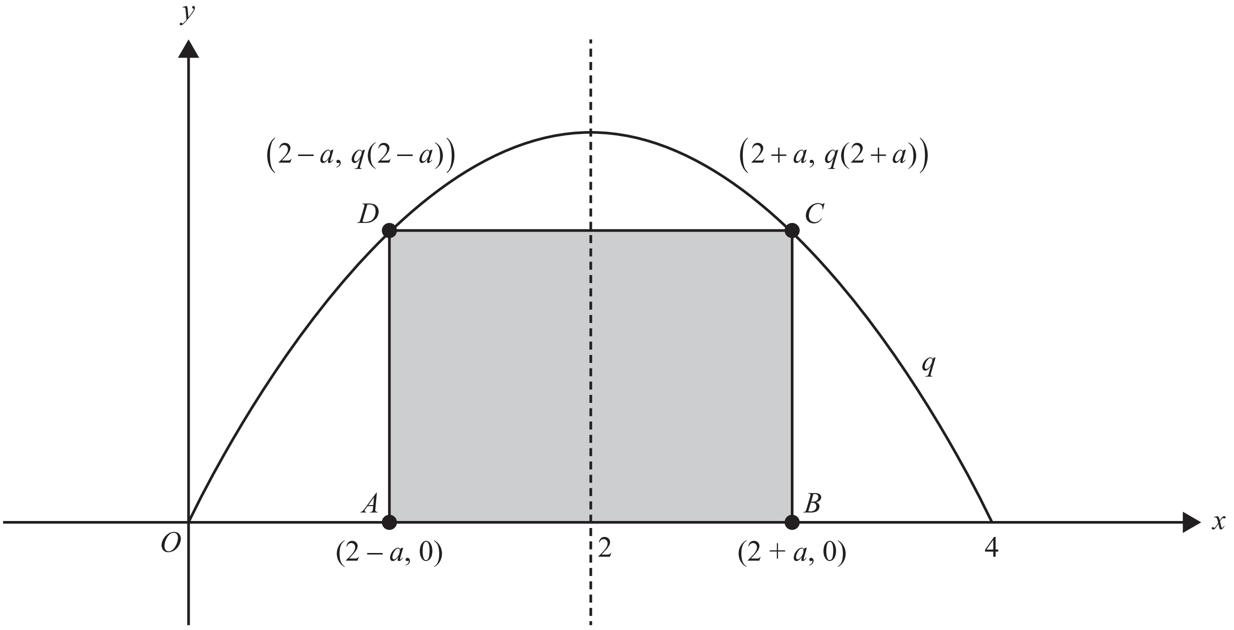
a. Find $\Pr(B)$. 2 marks

b. Find $\Pr(A' \cup B')$. 1 mark

Question 7 (5 marks)

Let $q : [0, 4] \to R$, $q(x) = x(4 - x)$.

A rectangle *ABCD* is inscribed between the graph of the function q and the x-axis. Its vertices are a units, where $a > 0$, from the axis of symmetry, $x = 2$, as shown below.



a. Find the value of a when the rectangle is a square. Give your answer in the form $b + \sqrt{c}$, where b is an integer and c is a positive integer. 2 marks

Question 7 – continued

b. Find the maximum area of the rectangle *ABCD*. Give your answer in the form $\dfrac{m\sqrt{n}}{p}$, where m, n and p are positive integers. 3 marks

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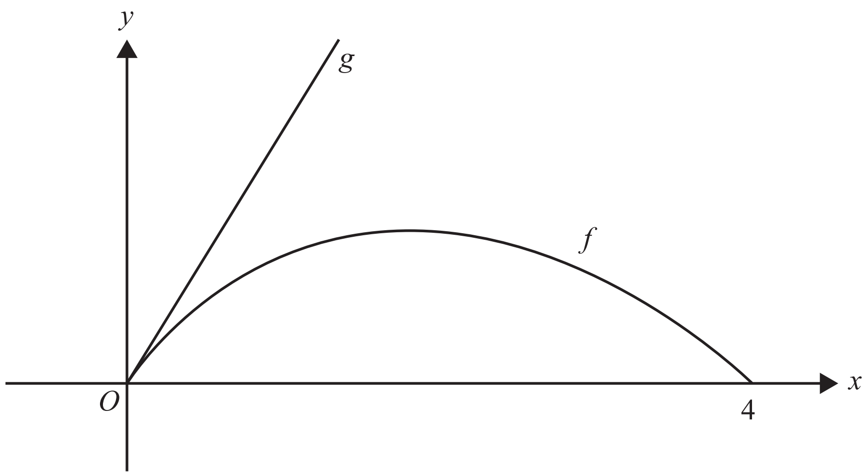
Question 8 (7 marks)

The graph of $f:[0,4]\to R,\ f(x)=x\left(2-\sqrt{x}\right)$ and part of the graph of $g:[0,\infty)\to R,\ g(x)=2x$ are shown below.



a. Find $f'(x)$. 1 mark

b. The tangent to the graph of f at the point $B\left(b, f(b)\right)$ is perpendicular to the graph of g.

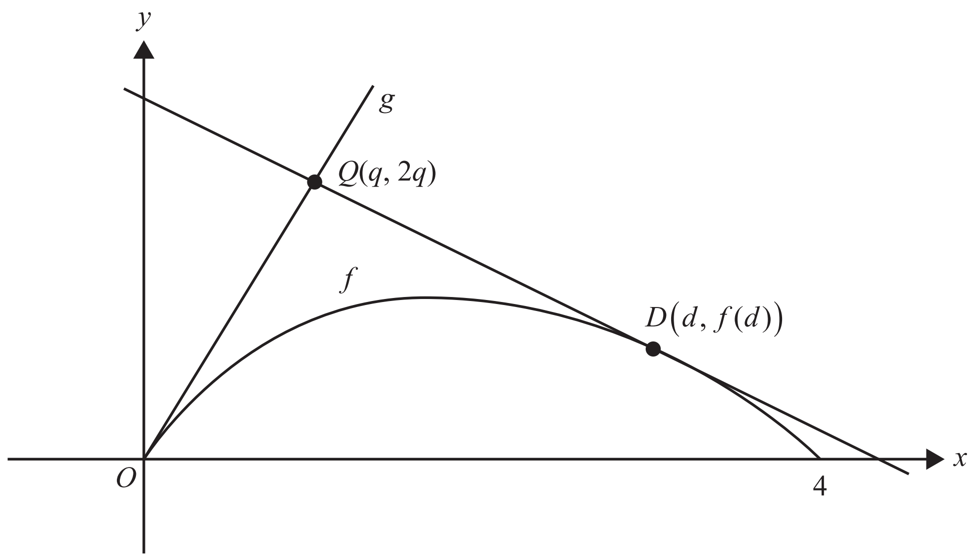
Find the coordinates of B. 3 marks

Question 8 – continued

c. Show that the graphs of f and g intersect only at the origin. 1 mark

d. Let $Q(q, 2q)$, where $q > 0$, be a point on the graph of g.

The tangent to the graph of f at the point $D(d, f(d))$ passes through Q, as shown below.



It can be shown that $d = 3q$.

Determine the values of q for which the tangent to the graph of f passes through Q and has an x-axis intercept greater than 4. 2 marks

Question 9 (3 marks)

A differentiable function $f : R \to R$ has the following two properties:

- $f'(x) = f(x)\big(4 - f(x)\big)$
- The range of f is $(0, 4)$.

a. Find $f'(0)$ if $f(0) = 1$. 1 mark

b. Determine, with appropriate justification, the number of stationary points of the graph of f. 1 mark

c. State the range of f'. 1 mark

END OF QUESTION AND ANSWER BOOK

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2021

MATHEMATICAL METHODS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference. A question and answer book is provided with this formula sheet.

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Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, \; n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c, \; x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a \; \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a \; \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a \; \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A\|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \text{E}(X)$	variance	$\text{var}(X) = \sigma^2 = \text{E}((X - \mu)^2) = \text{E}(X^2) - \mu^2$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x - \mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\text{E}(\hat{P}) = p$
standard deviation	$\text{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET