VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2023

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER									Letter

SPECIALIST MATHEMATICS

Written examination 1

Thursday 25 May 2023

Reading time: 10.30 am to 10.45 am (15 minutes)
Writing time: 10.45 am to 11.45 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 15 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (3 marks)

If $\sin(x) = 3\cos(x)$, find the value of $\sin(2x)$, where $x \in \left(0, \frac{\pi}{2}\right)$.

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Question 2 (4 marks)

The time taken for Sam to vacuum the house, V hours, is normally distributed with a mean of 1 hour and a standard deviation of 0.3 hours. The time taken for Sam to mop the floor, M hours, is independent of the time taken to vacuum and is normally distributed with a mean of 2 hours and a standard deviation of 0.4 hours. The time taken for Sam to clean the house, C hours, is the sum of the times taken to vacuum and to mop, $V + M = C$.

a. Find the expected value of C. 1 mark

b. Show that the variance of C is 0.25 1 mark

c. Sam wants to clean the house before a friend visits at 1.00 pm.

If Sam starts cleaning at 9.00 am, what is the probability that the cleaning will not be finished by 1.00 pm? Use $\Pr(-2 \le Z \le 2) = 0.9545$ and give your answer correct to three decimal places. 2 marks

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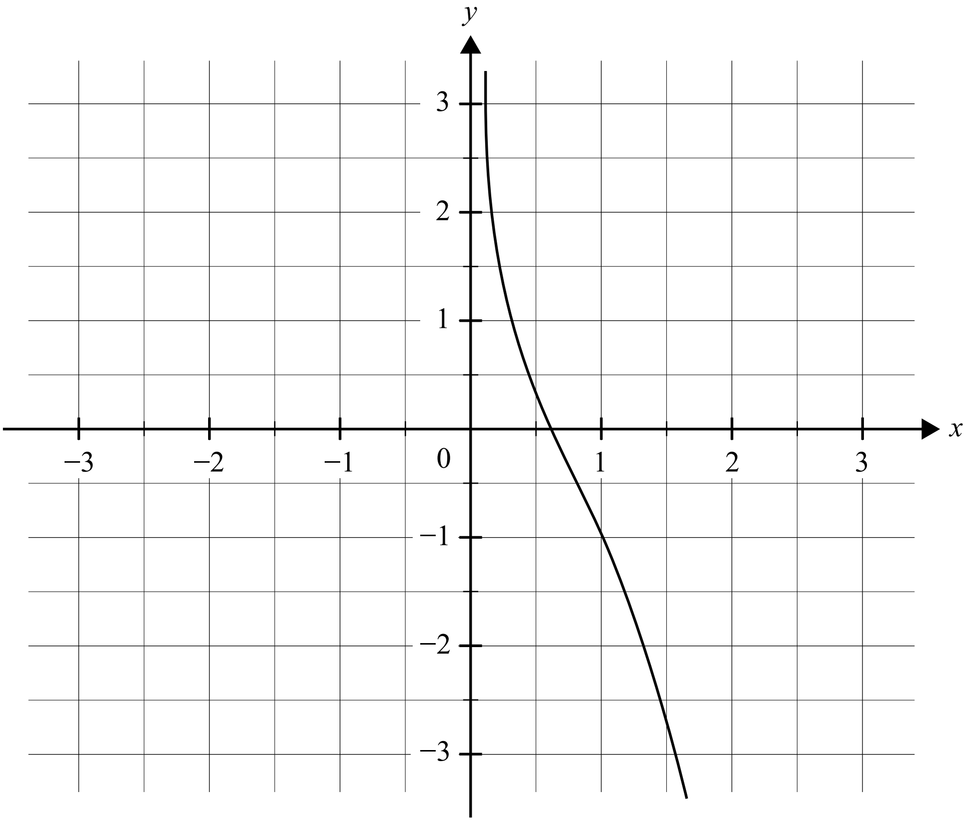
Question 3 (5 marks)

Part of the graph of f with rule $f(x)=\frac{1}{3x}-\frac{4x^2}{3}$ is shown below where $x>0$.



a. **i.** Find the second derivative of $f(x)$. 1 mark

ii. Hence, find the coordinates of the point of inflection. Label the point of inflection, with its coordinates, on the graph above. 2 marks

Question 3 – continued

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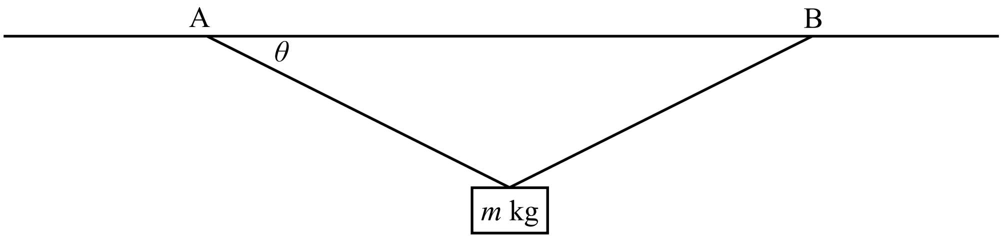
b. Complete the graph of f on the axes on page 6 by sketching the graph of f where $x < 0$. Find the coordinates of the stationary point and the equation of the vertical asymptote. Label both on the graph on page 6. 2 marks

Question 4 (3 marks)

The ends of a light inextensible string are fixed to a horizontal ceiling at points A and B, which are four metres apart. A body of mass m kilograms is attached to the middle of the string and hangs one metre below the midpoint of the line segment AB. The ceiling and the string meet at the acute angle θ.

a. Mark and label all forces acting on the mass on the diagram below. 1 mark



b. Given that the tension in the string is 5 N, find the value of m. 2 marks

Question 5 (4 marks)

a. Find the three solutions of the equation $z^3 = -8i$, where $z \in C$. Give your answers in Cartesian form. 2 marks

b. Let z_1 and z_2 be two of the solutions of the equation $z^3 = -8i$.

Show that the length of the straight line segment between the points represented by z_1 and z_2 is $2\sqrt{3}$. 1 mark

c. Hence, or otherwise, determine the area of the triangle whose vertices are the points representing the solutions of the equation $z^3 = -8i$. 1 mark

Question 6 (3 marks)

A function *f* has rule $f(x) = 5 + 2x^{\frac{3}{2}}$. The length of the curve for the graph of *f* in the interval $0 \le x \le k$ is $\dfrac{1022}{27}$.

Find the value of *k*.

Question 7 (5 marks)

A particle falls from rest such that its acceleration, in ms^{-2}, is given by $a = g - 2v$, where v is the velocity of the particle, in ms^{-1}, at time t seconds.

a. Solve a suitable differential equation to show that $v = \frac{g}{2}\left(1 - e^{-2t}\right)$. 4 marks

b. Write down the limiting (terminal) velocity of the particle in ms^{-1}. 1 mark

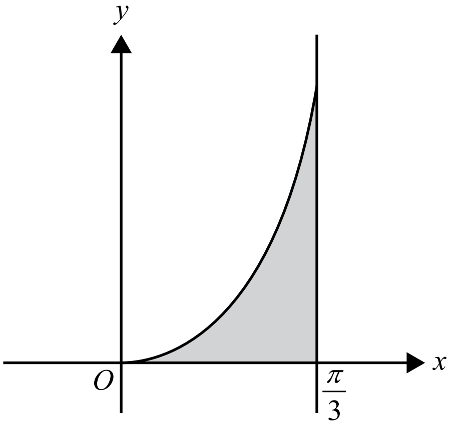
DO NOT WRITE IN THIS AREA

Question 8 (4 marks)

a. Show that $\frac{dy}{dx} = \sec(x)$, given that $y = \log_e\left(\sec(x) + \tan(x)\right)$. 2 marks

Question 8 – continued

b. Part of the curve with equation $y = \dfrac{1}{\cos(x)} - 1$ is shown below.



Find the shaded area, which is bounded by the curve, the x-axis and the line $x = \dfrac{\pi}{3}$. 2 marks

Question 9 (5 marks)

The position vectors of two cyclists, A and B, after t seconds are given by $\underset{\sim}{r}_A = (6t + a)\underset{\sim}{i} + (7t + 2 - 3a)\underset{\sim}{j}$ and $\underset{\sim}{r}_B = (5t + 4a)\underset{\sim}{i} + (6t + 2a - 2)\underset{\sim}{j}$ respectively for $t \geq 0$, where a is a positive constant and the components are measured in metres. The distance between the cyclists is initially $6\sqrt{2}$ metres.

a. Find a quadratic equation in the form $f(a) = 0$ that, when solved, gives the value of a. A solution to this equation is not required. 2 marks

b. Another cyclist, C, is moving with velocity given by $\dot{\underset{\sim}{r}}_C = 2t\underset{\sim}{i} + (n - 3)\underset{\sim}{j}$, where $n \in R$, with initial position $\underset{\sim}{r}_C = 2\underset{\sim}{i}$.

Given that $a = 2$, find the value of n such that the three cyclists, A, B and C, meet at the same point at the same time. 3 marks

Question 10 (4 marks)

Using a suitable substitution, evaluate $\int_{e}^{e^2} \left(\frac{\log_e\left(\log_e(x)\right)}{x\log_e(x)} \right) dx$.

END OF QUESTION AND ANSWER BOOK

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2023

SPECIALIST MATHEMATICS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference. A question and answer book is provided with this formula sheet.

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Specialist Mathematics formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
volume of a sphere	$\frac{4}{3}\pi r^3$
area of a triangle	$\frac{1}{2}bc\sin(A)$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	
$\sin(2x) = 2\sin(x)\cos(x)$	$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

Circular functions – continued

Function	$\sin^{-1}$ or arcsin	$\cos^{-1}$ or arccos	$\tan^{-1}$ or arctan
Domain	$[-1, 1]$	$[-1, 1]$	R
Range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r\left(\cos(\theta) + i\sin(\theta)\right) = r\,\text{cis}(\theta)$	
$\lvert z \rvert = \sqrt{x^2 + y^2} = r$	$-\pi < \text{Arg}(z) \le \pi$
$z_1 z_2 = r_1 r_2 \,\text{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}\left(\theta_1 - \theta_2\right)$
$z^n = r^n \,\text{cis}(n\theta)$ (de Moivre's theorem)	

Probability and statistics

for random variables X and Y	$\text{E}(aX + b) = a\text{E}(X) + b$ $\text{E}(aX + bY) = a\text{E}(X) + b\text{E}(Y)$ $\text{var}(aX + b) = a^2\text{var}(X)$
for independent random variables X and Y	$\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y)$
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$
distribution of sample mean $\overline{X}$	mean $\text{E}\left(\overline{X}\right) = \mu$ variance $\text{var}\left(\overline{X}\right) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e \lvert x \rvert + c$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$	$\int \sec^2(ax)\, dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c, a > 0$
$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c, a > 0$
$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$
	$\int (ax+b)^{-1} dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
Euler's method	If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$
acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
arc length	$\int_{x_1}^{x_2} \sqrt{1+\left(f'(x)\right)^2}\, dx$ or $\int_{t_1}^{t_2} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$
$\lvert \underset{\sim}{r} \rvert = \sqrt{x^2 + y^2 + z^2} = r$
$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$

Mechanics

momentum	$\underset{\sim}{p} = m\underset{\sim}{v}$
equation of motion	$\underset{\sim}{R} = m\underset{\sim}{a}$

END OF FORMULA SHEET